Abstracts

Double Phase Problems: From Musielak–Sobolev Spaces to Variational and Qualitative Methods

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This course offers a structured introduction to the analysis of **double phase problems**, one of the central themes in the contemporary study of nonlinear elliptic equations with **non-standard growth**. Such problems arise from variational integrals whose density changes its behavior from one power law to another, typically according to the prototype

$$\mathcal{F}(u) = \int_{\Omega} \left(|\nabla u|^p + a(x)|\nabla u|^q \right) dx, \qquad 1 < p < q,$$

where the coefficient $a(x)$ governs the transition between the two phases. This framework is closely related to questions from the calculus of variations, regularity theory, anisotropic media, and the general theory of partial differential equations in generalized Orlicz and Musielak-type spaces.

The course is organized into **two main parts**.

The **first part** (May 4-15) is devoted to the functional setting underlying the theory. We begin with modular methods and introduce Lebesgue, Orlicz, and Musielak–Orlicz type spaces, with particular emphasis on Musielak–Lebesgue and Musielak–Sobolev spaces as the natural environment for double phase operators. We discuss modulars, Luxemburg norms, convergence, separability, reflexivity, duality, uniform convexity, density of smooth functions, Poincaré-type inequalities, and embedding theorems. The purpose of this first part is not only to present abstract definitions, but also to explain why these spaces are the correct analytic framework for nonhomogeneous and nonuniformly elliptic problems.

The **second part** (June 15-25) focuses on the PDE and variational side of the subject. After introducing the double phase energy, the associated Euler–Lagrange equations, and the notion of weak solution, we present the main tools used in current research. These include direct methods in the calculus of variations, coercivity and lower semicontinuity arguments, monotonicity methods, mountain pass geometry, Nehari manifold techniques, fibering maps, eigenvalue-type problems, and sub- and supersolution methods. We also discuss selected qualitative issues such as multiplicity of solutions, sign properties, boundedness, and the role played by the interaction between the exponents and the coefficient $a(x)$. Depending on time, the course may also touch on regularity questions and on recent developments involving variable exponents or anisotropic variants.

The course is designed both as an advanced introduction and as a bridge toward the current literature. A central feature of the course is the constant interplay between the abstract functional framework and concrete nonlinear problems: the first part develops the space theory needed later, while the second part shows how that theory is used in existence, multiplicity, and qualitative analysis for double phase equations.