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STOCHASTIC CONTROL OF AN SDOF SYSTEM UNDER COMBINED PERIODIC AND WHITE NOISE EXTERNAL EXCITATIONS

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Abstract. *The paper considers a problem of stochastic control and dynamics of a single-degree-of-freedom system, subjected to combined periodic and white noise external excitations. To minimize the system response energy a bounded in magnitude control force is applied to the systems. The stochastic optimal control problem is handled through the dynamic programming approach. Based on the solution to the Hamilton-Jacobi-Bellman equation it is proposed to use the dry friction as a suboptimal control law. In the resonant case the stochastic averaging procedure has been used to derive stochastic differential equations for the system response amplitude and phase. The joint PDF of response amplitude and phase is derived by finding an exact analytical solution to the corresponding Fokker-Plank-Kolmogorov equation for the resonant case. The Path Integration (PI) method is used to construct the joint response PDF for non-resonant cases.*

1 INTRODUCTION

There is a variety of engineering systems, where the external periodic excitation may be combined with random loading, which may not be neglected. Since the system becomes stochastic it requires stochastic control theory to establish the proper control strategy. Dynamic Programming (DP) approach1 provides such a technique, which enables to convert a problem of optimal control to a problem of finding solution to the corresponding multidimensional, in general nonlinear and degenerate partial differential equation of parabolic type for Bellman function – the Hamilton-Jacobi-Bellman (HJB) equation. The HJB equation must be satisfied within the entire state-space domain and the asymptotic behaviour of the Bellman function is unknown. These two facts make it practically impossible to apply any standard numerical technique to solve the HJB equation.

Except the linear quadratic regulator (LQR) problem, there have been no exact analytical solutions to a HJB equation for dynamic systems. Recently a new approach to problems with bounded in magnitude control force has been proposed2-4. The method suggests handling the problem by splitting it into two steps. At the first step an exact analytical solution to the HJB equation is found in the outer domain. At the second step the HJB equation is solved numerically within the bounded state-space domain, where the analytical solution is used as boundary conditions, thereby finding a solution to the HJB equation within the entire state space. It has been proven mathematically2 that the obtained analytical solution indeed describes the asymptotic behaviour of the Bellman function, hence it can be used as boundary conditions.

Finding and applying the optimal control strategy to the system makes it, in general, nonlinear. Analysis of a nonlinear system subjected to combined periodic and white noise external excitations may be handled by the stochastic averaging procedure5, 6. It is common to use a standard transformation of variables to derive stochastic differential equations with respect to slowly varying amplitude and phase of the response. Another transformation of variables was proposed in7, which was used by different authors. The method of multiple scale along with closure technique were used to derive an analytical expression for the response of the Duffing-Rayleigh oscillator subjected to combined periodic and random external excitation8.

In the case of bounded in magnitude control, the dynamic system possesses a special type of nonlinearity – nonlinearity of signum type, thus the transformation, proposed in9, cannot be applied. Such a strong nonlinearity leads to the delta-function in the corresponding FPK equation, written for the system displacement and velocity. It has been shown that a special adaption of the PI numerical method10 is required to calculate the system response joint probability density function (PDF) if the system possesses the signum type nonlinearity.

In the paper a problem of stochastic optimal control of a single-degree-of freedom (SDOF) system is considered. A bounded in magnitude external control force is applied to the system to minimize quadratic cost function, which is equal to the system response energy in the special case. The DP approach is used to solve the problem of stochastic optimal control. To analyse the system the stochastic averaging procedure is used. In the resonant case an analytical solution to the corresponding FPK equation is derived. Results of analytical calculations are compared with the results of direct numerical simulation.

2 PROBLEM STATEMENT

Consider a dynamic system subjected to external white noise and periodic excitations. The governing equation of motion in a state-space form may be written as:

$$\begin{gathered} \dot{x}_1 = x_2, \\ \dot{x}_2 = -2\alpha x_2 - \Omega^2 x_1 + v + \sigma\xi(t) + \lambda \sin(\omega t), \qquad 0 < t \le T, \\ x_1(0) = x_{10}, x_2(0) = x_{20}, |v(t)| \le R, E[\xi(t)\xi(t+t_0)] = \delta(t_0) \end{gathered} \tag{1}$$

where λ is the excitation amplitude, ω is the excitation frequency and $\xi(t)$ is the zero-mean Gaussian white noise. Assume that the aim of the control is to minimize the following quadratic functional, which is equal to the system mean response energy when a = 1 or b = 1:

$$J_{x_1,x_2,t}(v) = E\left\{\frac{a}{2}[\Omega^2 x_1^2(T) + x_2^2(T)] + \int_t^T \frac{b}{2}[\Omega^2 x_1^2(s) + x_2^2(s)]\,ds\right\} \tag{2}$$

Then, the Bellman function:

$$u(x_1, x_2, t) = inf\{J_{x_1,x_2,t}(v):\ |v| \le R\} \tag{3}$$

must satisfy the following HJB equation:

$$\frac{\partial u}{\partial t} + Lu + \lambda \sin(\omega t)\frac{\partial u}{\partial x_2} + inf_{|v|\le R}\left\{v\frac{\partial u}{\partial x_2}\right\} + g = 0 \tag{4}$$

where:

$$Lu = x_2\frac{\partial u}{\partial x_1} - (2ax_2 + \Omega^2 x_1)\frac{\partial u}{\partial x_2} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x_2^2}, \qquad g(x_1, x_2) = \frac{b}{2(\Omega^2 x_1^2 + x_2^2)} \tag{5}$$

Introducing the inverse time $\tau = T - t$ and evaluating the optimal control gives:

$$v = -Rsign\left(\frac{\partial u}{\partial x_2}\right), inf_{|v|\le R}\left\{v\frac{\partial u}{\partial x_2}\right\} = -R\left|\frac{\partial u}{\partial x_2}\right| \tag{6}$$

The following Cauchy problem is formulated:

$$\frac{\partial u}{\partial \tau} = Lu + \lambda sin[\omega(T-\tau)]\frac{\partial u}{\partial x_2} - R\left|\frac{\partial u}{\partial x_2}\right| + g; \qquad u(x_1, x_{2,0}) = \frac{\alpha}{2}(\Omega^2 x_1^2 + x_2^2) \tag{7}$$

It can be seen that the equation (7) represents a nonlinear, degenerate PDE of parabolic type, which has to be satisfied within the whole state-space domain. The solution to HJB equation (7) provides an optimal control strategy, defined in (6).

3 SOLUTION TO THE HJB EQUATION

According to the proposed methodology[4], assume that there is a domain of state-space where $z = sign\left(\frac{\partial u}{\partial x_2}\right)$ stays constant for all values of $x_1, x_2, \tau > 0.$ Then, within this domain the following *modified* HJB equation is valid:

$$\frac{\partial u}{\partial \tau} = Lu + \lambda sin[\omega(T-\tau)]\frac{\partial u}{\partial x_2} - Rz\frac{\partial u}{\partial x_2} + g \tag{7}$$

Let us use the quadratic function approach, proposed earlier[4], to obtain the solution to the modified HJB equation (7):

$$u(\tau, x_1, x_2) = \sum_{i,j=0}^{2} f_{ij}(\tau)x_i x_j = f_{00}(\tau) + f_{10}(\tau)x_1 + f_{20}(\tau)x_2 + f_{12}(\tau)x_1 x \tag{8}$$

Substitution of (8) into (7) results in a set of ODEs for coefficients $f_{ij}(\tau) = f_{ij}$. Solving these equations, as presented in[4], is valid in the outer domain defined as:

$$|x_2| \ge \frac{\lambda\omega}{|\Omega^2 - \omega^2|} \tag{9}$$

The inner domain, supplementary to the outer domain (9), forms a strip of a finite width in x_2 direction and infinite in $\pm x_1$ direction. Within the outer domain the dry friction $v = Rsign(x_2)$ provides the optimal control law. It can be seen from (9) that the width of the inner domain will reduce with larger difference between excitation and natural frequencies. In the limiting case when this difference is very large, the inner domain would reduce to the axis x_2 and the dry friction law becomes optimal within the entire state-space. This result is important since it excludes the necessity of solving the corresponding HJB equation. Moreover, it resembles the case of pure external random excitation3. Thus for the large values of detuning the influence of periodic excitation in terms of control strategy is negligible. In this case the dry friction law may be considered as suboptimal. In the next section, this preposition is directly verified by comparison with the results of numerical simulations.

4 DYNAMICS OF THE CONTROLLED SYSTEM

Application of the control force $v = Rsign(\dot{x})$ to the system leads to the following nonlinear SDE:

$$\begin{gathered} \dot{x}_1 = x_2, \\ \dot{x}_2 = -2\alpha x_2 - \Omega^2 x_1 - Rsign(x_2) + \sigma\xi(t) + \lambda\sin(\omega t), \qquad 0 < t \leq T, \\ x_1(0) = x_{10}, x_2(0) = x_{20} \end{gathered} \tag{10}$$

To use the stochastic averaging procedure, the standard transformation of variables is applied:

$$x(t) = A(t)\cos\theta(t), \dot{x}(t) = -A(t)\sin\theta(t), \theta(t) = \omega t + \varphi(t) \tag{11}$$

where $A(t)$ and $\varphi(t)$ are the slowly varying amplitude and phase processes respectively. Differentiating (11) with respect to time, multiplying by $\cos\theta$ and $\sin\theta$ and assuming that each term on the right hand sides of the resulting equations is proportional to a small parameter, both equations can be averaged over the period:

$$\begin{gathered} dA = \left[-\alpha A - \frac{2R}{\omega\pi} + \frac{\sigma^2}{4\omega^2 A} - \frac{\lambda}{2\omega}\cos\phi\right]dt - \frac{\sigma}{\sqrt{2}\omega}dB_1 \\ d\phi = \left[-\frac{\Delta}{2\omega} + \frac{\lambda}{2\omega A}\sin\phi\right]dt - \frac{\sigma}{\sqrt{2}\omega A}dB_2 \end{gathered} \tag{12}$$

where B_1 and B_2 are uncorrelated Wiener processes with zero mean and $\Delta = \omega^2 - \Omega^2$.

From the solution to the HJB equation in the non-resonant case and from the discussion by other authors it follows that the influence of the periodic excitation in this case is small and the process tends to be Markovian. Thus, for large values of detuning the external excitation in (12) may be disregarded, assuming $\lambda = 0$. Then the equations (12) become uncoupled and the steady-state mean $E[A] = m_\alpha$ and mean square value of the amplitude $E[A^2] = D_A$ may be found from the solution to the corresponding FPK equation or directly from (12):

$$\dot{D_A} = 0 \Rightarrow 2\alpha D_A + \frac{4R}{\omega\pi}m_\alpha = \frac{\sigma^2}{\omega^2} \tag{13}$$

In the case of $\alpha = 0$ the mean value of amplitude is found as:

$$m_\alpha = \frac{\sigma^2\pi}{4R\omega} \tag{14}$$

The formula (14) is in a good agreement with the results of numerical simulations for relatively high excitation frequency: $\omega/\Omega > 2$. It should be stressed that for numerical simulation the values of R and λ should be taken proportional to a small parameter as it is required by the stochastic averaging approach. However, the results of the extensive numerical simulations for other, not necessarily small, values of R and λ have shown the similar behaviour in the vicinity of and far from the resonance.

In the resonant case $(\Delta = 0)$ the stationary FPK equation may be written, rearranging the derivatives, as:

$$\frac{\partial}{\partial A}\left(\left[-\alpha A - \frac{2R}{\omega\pi} + \frac{\sigma^2}{4\omega^2 A} - \frac{\lambda}{2\omega}\cos\phi\right]p - \frac{\sigma^2}{4\omega^2}\frac{dp}{dA}\right) + \frac{\partial}{\partial x}\left(\left[\frac{\lambda}{2\omega A}\sin\phi\right]p \tag{15}$$

Equating each of the expressions under derivative to zero, integrating the first equation with respect to A we obtain and substituting this expression into the second equation gives the identity, where C is normalization constant. The solution may be simplified using the property of logarithm, namely:

$$p(A,\phi) = CAexp\left\{\delta\left[-\frac{\alpha A^2}{2} - A(\Delta_1 + \Delta_2\cos\phi)\right]\right\},$$
$$\delta = \frac{4\omega^2}{\sigma^2}, \Delta_1 = \frac{2R}{\omega\pi}, \Delta_2 = \frac{\lambda}{2\omega}, \delta\Delta_1 = \frac{2}{m_\alpha} \tag{16}$$

Consider the case of $\alpha = 0$, which may be of special interest, since in this case the dynamic system has no means to dissipate the energy other than due to the control force. In this case for stability in probability we obtain $\mu = \frac{4R}{\lambda\pi} > 1$. It is interesting to note that in the resonant case the purely deterministic system $(\xi = 0)$ is unstable when $\alpha = 0$. Calculating the integration constant leads to expressions for the probability density of the phase process and the mean amplitude which are given by:

$$p(\phi) = \frac{2\left(1 - \frac{1}{\mu^2}\right)^{\frac{3}{2}}}{\pi\left[1 + \left(\frac{1}{\mu}\right)\cos\phi\right]^2}, \tag{17}$$

$$E[A] = C\int_0^{2\pi}\int_0^{+\infty} A^2 exp[-A\delta(\Delta_1 + \Delta_2\cos\phi)]dAd\phi = \frac{m_\alpha\left(2 + \frac{1}{\mu^2}\right)}{2\left(1 - \frac{1}{\mu^2}\right)} \tag{18}$$

In the limiting case of $\lambda \to 0, \mu \to \infty$ formula (18) tends to $E[A] = m_\alpha$ as expected. The results show that the ratio $\frac{E[A]}{m_\alpha}$ is far from unity when the system is near its stability boundary, i.e. $\mu \approx 1$.

5 RELIABILITY OF THE CONTROLLED SYSTEM

Here we study one possible way of the system's failure – the first passage problem. It is associated with the system's response reaching its critical values, which leads to immediate system failure. The first-passage problem[5] may be formulated as a problem of finding a time t_c when the system overcomes its threshold value for the first time. Since the excitation process is random, we are interested in the mean value of the time $T = E[t_c]$, which can be found as a solution to the corresponding Pontryagin equation[5]. Equation (12) may be written as:

$$\left[-\alpha A-\frac{2R}{\omega\pi}+\frac{\delta}{A}-\frac{\lambda}{2\omega}\cos\phi\right]\frac{\partial T}{\partial A}+\delta\frac{\partial^2 T}{\partial A^2}=-1, \tag{19}$$

with conditions $T'(A)\approx o(A), A\to 0 \; and \; T(A_c)=0$ (20)

As it can be seen, equation (19) contains the response phase. In the non-resonant case the deterministic excitation will not significantly influence the response, whereas in the resonant case the influence of the deterministic excitation is significant. Moreover, the phase of the deterministic system at the resonance may be equal to either zero or π. Thus, we consider the resonant case and take $\varphi = \pi$. It is possible to derive analytical solution for the derivative of the mean time by integrating equation (19) once:

$$T'(A)=\frac{h}{A}\sqrt{\frac{\pi}{2\delta\alpha^3}}e^{r^2(A)}(\Phi[r(A)]-\Phi[r(0)])+\frac{1}{\alpha A}\left[1-exp\left(\frac{\alpha A^2+2Ah}{2\delta}\right)\right] \tag{21}$$

where $r(A)=\frac{\alpha A+h}{\sqrt{2\delta\alpha}}, h=\frac{2R}{\omega\pi}-\frac{\lambda}{2\omega}$ (22)

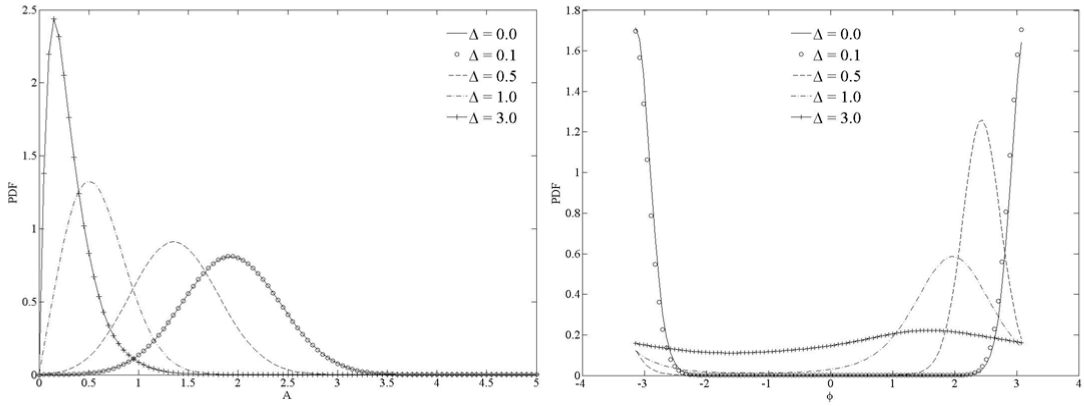
and Φ – the error function. It is impossible to integrate expression (21) analytically; therefore it has to be done numerically. The numerical integration has shown that the results significantly depend on the values of noise intensity σ and h. Clearly, increase of noise intensity decreases the mean first passage time of the system. Similar trend is observed when h is decreasing, which is expected since this leads the system to its instability boundary.

6 PATH INTEGRATION FOR NON-RESONANT CASES

In the case away from resonance, the PI method[10] is used to derive the joint response PDF of system (10). This is a numerical iterative approach based on the Markovian nature of the response, solving the Chapman-Kolmogorov equation:

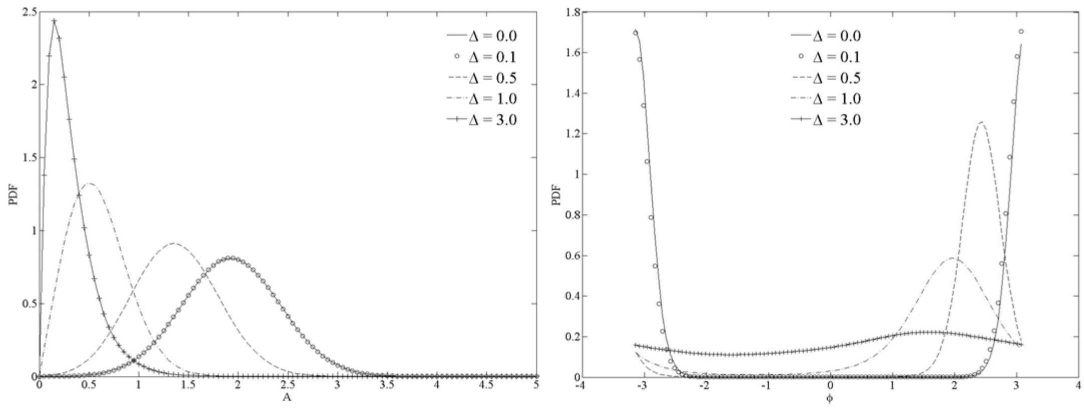
$$p(x,y,t)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}p(x,y,t|x',y',t')p(x',y',t')dxdy \tag{23}$$

where the prime denotes the previous time step. At this point is worth separating this numerical approach from other analytical path integration methods e.g., a Wiener path integral technique[11]. Starting from an initial density function and applying the transitional probability density function to it the joint response PDF is acquired. This method has been proved to be very efficient for constructing response PDFs of highly nonlinear and parametric systems[9, 12]. Figure 1 and Figure 2 depict the PDFs of the system's response for $\sigma^2=0.1, \lambda=0.5$, two cases of damping and control force and several detuning values including resonance. In both figures, it is noticed that larger detuning Δ concentrates most of the probability at lower amplitude values, as expected even from classical dynamics theory. As suggested by equations (14) and (18), the same effect on amplitude stands for the control force R too when comparing Fig. 1(a) and 2(c). Regarding the phase process, in the resonance case it tends to uniform distribution as $R\to\infty$, as suggested by equation (17).



(a) (b)

Figure 1: PDFs for (a) the amplitude and (b) phase for R=0.1 and α=0.1 and different detuning values.



(c) (d)

Figure 2: PDFs for (a) the amplitude and (b) phase for R=0.5 and α=0.0 and different detuning values.

7 CONCLUSIONS

In this paper a problem of stochastic optimal control of an SDOF system subjected to combined deterministic and white noise external excitations is considered. It has been shown that in the non-resonant case the dry friction provides the suboptimal control law and the system response is not significantly influenced by the deterministic excitation. In the resonance case the control strategy is significantly different from the dry friction law. An exact analytical solution of the corresponding FPK equation is derived in this special case. It has been shown that unlike the purely deterministic system, which is unstable when the dry friction law is implemented, the system with the combined excitation may be stable providing that $\mu > 1$. The probability density function of the phase process $\varphi(t)$ is not uniformly distributed, but it tends to this distribution as the value of dry friction coefficient R tends to infinity. The analytical result for a value of mean response amplitude $A(t)$ is in a very good agreement with the results of numerical simulations.

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