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LINEAR AND NON-LINEAR SYSTEMS UNDER SUB-GAUSSIAN (α-STABLE) INPUT

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Abstract. *The paper deals with the analysis of linear and non-linear systems under a special class of symmetric α-stable stochastic processes, namely sub-Gaussian excitations. Such processes are defined multiplying the square root of an $\alpha/2$-stable random variable totally skewed to the right by a zero mean normal process with assigned autocorrelation function. Relying on the observation that the sub-Gaussian input may be viewed as a Gaussian process with random amplitude having $\alpha/2$-stable distribution, it is shown that the characteristic function and the probability density function of the response can be obtained from those of the system subject to the underlying Gaussian process by performing simple integrals. It is also observed that linear systems are amenable to closed-form solutions in terms of characteristic function of the response. Appropriate comparisons with the exact solutions and Monte Carlo simulation results demonstrate the accuracy of the procedure in the linear and non-linear cases, respectively.*

Sommario. *Oggetto del presente lavoro è l'analisi di sistemi lineari e non-lineari soggetti a una particolare classe di processi aleatori simmetrici α-stabili, noti come processi sub-Gaussiani. Tali processi sono definiti moltiplicando la radice quadrata di una variabile aleatoria $\alpha/2$-stabile totalmente deviata a destra per un processo aleatorio Gaussiano a media nulla di assegnata funzione di autocorrelazione. Osservando che una forzante sub-Gaussiana può essere considerata come un processo Gaussiano caratterizzato da un'ampiezza aleatoria avente distribuzione $\alpha/2$-stabile, viene mostrato che la funzione caratteristica e la funzione densità di probabilità della risposta possono essere ottenute a partire da quelle del sistema soggetto al processo Gaussiano di base mediante il calcolo di* *integrali semplici. Si osserva, inoltre, che per sistemi lineari la funzione caratteristica della risposta può essere determinata in forma chiusa. L'accuratezza della procedura è dimostrata mediante opportuni confronti con le soluzioni esatte nel caso lineare e con i risultati della simulazione Monte Carlo nel caso di sistemi non-lineari.*

1 INTRODUCTION

The Central Limit Theorem (CLT) justifies the extensive use of Gaussian processes for modeling a wide class of physical phenomena. Well-established techniques for the analysis of linear and non-linear systems under Gaussian input may be found in classical textbooks on random vibration theory[1-5]. In this context, a central role is played by the Gaussian white noise process, formal derivative of the so-called Wiener process. Indeed, the special features of the Wiener process enable to perform the probabilistic characterization of the response to Gaussian white noise input by using the powerful tools of Itô stochastic differential calculus[6]. However, many real phenomena observed in physics, seismology, electrical engineering, economics, etc., are non-Gaussian in nature, as they belong to a heavy-tailed distribution class or have impulsive nature. The necessity of non-Gaussian models for describing the large fluctuations exhibited by such phenomena have raised an increasing interest in the so-called *α-stable Lévy processes*[7]. Such processes are characterized by four parameters: the *stability index* $\alpha \in (0,2]$, the *scale* $\sigma > 0$, the *skewness* $\beta \in [-1,1]$ and the *shift* $\mu \in \mathbb{R}$. An appropriate selection of the parameters α, σ, β and μ yields a rich variety of α-stable Lévy noises, which may be adequately used to model various phenomena such as income distributions in economics, seismic ground acceleration in earthquake engineering, gravitational forces acting on stars, etc. The Gaussian white noise is a special case of the α-stable Lévy white noise for $\alpha = 2$. The response of linear and non-linear systems driven by α-stable Lévy white noises has been widely investigated in the literature[8-23] working either in terms of Probability Density Function (PDF), ruled by the Einstein-Smoluchowsky (ES) equation[16,17], or of Characteristic Function (CF). Grigoriu[11] obtained closed-form solutions for the CF and the mean up-crossing rate of the response of linear systems to α-stable processes based on the integral and series representation of the input process. Non-linear systems have been handled by different approaches such as digital simulation[10,12,14], path integral method[12], equivalent linearization technique[17] and wavelet expansion[23]. Closed-form expressions of the PDF or CF can be found only in the stationary case for some scalar systems with polynomial nonlinearities[18,19].

This paper is devoted to the analysis of linear and non-linear systems under a special class of symmetric α-stable processes, namely sub-Gaussian excitations. Such processes are defined as the product of a zero mean Gaussian process $G(t)$, having assigned autocorrelation function, and the square root of an $\alpha/2$-stable random variable ($\alpha < 2$), A, totally skewed to the right ($\beta = 1$) and independent of $G(t)$. The sub-Gaussian process $A^{1/2}G(t)$ is also called *subordinate* to the underlying Gaussian process $G(t)$. In the linear case, closed-form solutions in terms of CF can be easily obtained by taking into account that the response process to the above defined input is sub-Gaussian too. The procedure presented in the paper relies on the observation that the subordinate input $A^{1/2}G(t)$ may be viewed as a conditional Gaussian process, namely as a Gaussian process with random amplitude having $\alpha/2$-stable distribution. Specifically, it is shown that the probabilistic characterization of the response of dynamic systems under sub-Gaussian input can be performed starting from the response statistics of the system subject to the underlying Gaussian process. In this regard, it has to be mentioned that, while closed-form solutions are always available for the PDF and CF of the response of linear systems under Gaussian input, in the non-linear case the evaluation of response statistics is not an easy task. Nevertheless, if the underlying Gaussian process is a white noise, the PDF is known in explicit form for some special classes of non-linear systems[5,26-28]. In the other cases, approximate solutions of the partial differential equation ruling the PDF (Fokker-Planck-Kolmogorov equation) or the CF can be obtained by any of the procedures developed in the literature.

Some numerical results concerning the response PDF and CF of one-dimensional linear and non-linear systems under α-stable sub-Gaussian input for various values of the stability index are presented in the paper. The accuracy of the proposed procedure is demonstrated through appropriate comparisons with the exact solution and Monte Carlo simulation (MCS) data in the linear and non-linear cases, respectively.

2 α-STABLE RANDOM VARIABLES AND PROCESSES

In this section, some basic concepts concerning α-stable random variables and processes are briefly summarized for clarity's sake as well as for introducing appropriate notations. The readers interested in this topic are referred among others to Samorodnitsky and Taqqu[7].

A random variable X is said to have a *stable* distribution if for any positive numbers m and n, there is a positive number p and a number $u \in \mathbb{R}$ such that:

$$mX_1 + nX_2 \overset{d}{=} pX + u \tag{1}$$

where X_1 and X_2 are independent copies of X and "$\overset{d}{=}$" denotes equality in distribution. The characteristic function (CF) of such random variables, denoted as $\phi_X(\vartheta)$, is given by:

$$\phi_X(\vartheta) = E[\exp(\mathrm{i}\vartheta X)] = \begin{cases} \exp\left\{-\sigma^\alpha |\vartheta|^\alpha \left[1 - \mathrm{i}\beta \operatorname{sign}(\vartheta) \tan\left(\dfrac{\pi\alpha}{2}\right)\right] + \mathrm{i}\mu\vartheta\right\}, \text{ if } \alpha \neq 1; \\ \exp\left\{-\sigma|\vartheta|\left[1 + \mathrm{i}\beta\dfrac{2}{\pi}\operatorname{sign}(\vartheta)\ln|\vartheta|\right] + \mathrm{i}\mu\vartheta\right\}, \text{ if } \alpha = 1 \end{cases} \tag{2}$$

where $E[\cdot]$ means stochastic average; $\mathrm{i} = \sqrt{-1}$ is the imaginary unit; $\alpha \in (0,2]$, $\sigma > 0$, $\beta \in [-1,1]$ and $\mu \in \mathbb{R}$ are four parameters commonly referred to as *stability index* (or *characteristic exponent*), *scale parameter* (or *dispersion*), *skewness* (or *asymmetry*) and *shift* (or *location*), respectively. The index of stability α describes the tails of the probability density, the parameters σ and β govern, respectively, the spread and skewness of the distribution around its center which is defined by the shift μ with respect to the origin. Random variables having a CF as in Eq. (2) are also called *α-stable random variables* and are denoted as $X \sim S_\alpha(\sigma,\beta,\mu)$. In the case in which $\beta = \mu = 0$, the random variable X is symmetric and is denoted as $X \sim S_\alpha S$. The probability densities of α-stable random variables exist and are continuous, but they are not always known in a simple explicit form. In fact, the Inverse Fourier Transform of the CF in Eq. (2) can be performed analytically only in few special cases: the *Gaussian* distribution $X \sim S_2(\sigma,0,\mu)$, the *Cauchy* distribution

$X \sim S_1(\sigma, 0, \mu)$ and the *Lévy* distribution $X \sim S_{1/2}(\sigma, 1, \mu)$. Specifically, for normal random variables ($\alpha = 2$) the scale parameter σ is proportional to the standard deviation σ_X ($\sigma_X = \sqrt{2}\sigma$), the skewness β can be taken to be zero and μ is the mean.

The striking feature of α-stable random variables is that their Probability Density Function (PDF) has inverse power tails, which implies that the tails decay more slowly than those of Gaussian distributions. Specifically, the rate of decay depends on the characteristic exponent α, in such a way that the smaller α heavier the tails. As a consequence, the variance and higher order statistical moments of α-stable random variables are infinite, with the exception of the case $\alpha = 2$ (Gaussian distribution). In fact, for $\alpha \in (0,2)$:

$$\begin{aligned} E\left[|X|^r\right] &= \infty, \quad \text{for } r \geq \alpha; \\ E\left[|X|^r\right] &< \infty, \quad \text{for } r \in (0,\alpha). \end{aligned} \tag{3}$$

Of course, when $\alpha \leq 1$ the mean is infinite as well.

A random variable $X = A^{1/2}G$ is $S_\alpha S$ ($\alpha < 2$) if G has a zero mean Gaussian distribution, $G \sim S_2(\sigma, 0, 0)$, and A is an $\alpha/2$-stable random variable totally skewed to the right, $A \sim S_{\alpha/2}((\cos(\pi\alpha/4))^{2/\alpha}, 1, 0)$, and independent of G. Such random variables are also called *sub-Gaussian* or *subordinate* to G and have the following CF:

$$\phi_X(\vartheta) = E[\exp(\mathrm{i}\vartheta X)] = \exp\left\{-\left|\frac{1}{2}\vartheta^2\sigma_G^2\right|^{\alpha/2}\right\} \tag{4}$$

where σ_G denotes the standard deviation of G. It can be observed that each $S_\alpha S$ random variable is conditionally Gaussian[7], namely $X = A^{1/2}G$ may be viewed informally as normal with the random variance $\sigma_G^2 A$.

Similarly, a random vector $\mathbf{X} \in \mathbb{R}^n$ is called *sub-Gaussian* $S_\alpha S$ ($\alpha < 2$) with underlying Gaussian vector $\mathbf{G}$ or *subordinate* to $\mathbf{G}$ if it is defined as:

$$\mathbf{X} = A^{1/2}\mathbf{G} \tag{5}$$

being $A \sim S_{\alpha/2}((\cos(\pi\alpha/4))^{2/\alpha}, 1, 0)$ independent of $\mathbf{G}$. The CF of the random vector $\mathbf{X}$ defined in Eq. (5) takes the following form:

$$\phi_{\mathbf{X}}(\boldsymbol{\vartheta}) = E[\exp(\mathrm{i}\boldsymbol{\vartheta}^{\mathrm{T}}\mathbf{X})] = \exp\left\{-\left|\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\vartheta_j\vartheta_k\sigma_{jk}^2\right|^{\alpha/2}\right\} \tag{6}$$

where $\sigma_{jk}^2 = E[G_j G_k]$.

An α-stable stochastic process $\{X(t),\ t \in T\}$ may be defined as an α-stable random variable depending on the parameter set T.

Let $\{G(t),\ t \in T\}$ be a zero mean Gaussian process and let A be an $\alpha/2$-stable random variable $A \sim S_{\alpha/2}((\cos\pi\alpha/4)^{2/\alpha}, 1, 0)$, ($\alpha < 2$), independent of $G(t)$, then $X(t) = A^{1/2}G(t)$ is a *sub-Gaussian process* whose CF is given by:

$$\phi_{\mathbf{X}}(\boldsymbol{\vartheta};\mathbf{t}) = E[\exp(\mathrm{i}\boldsymbol{\vartheta}^{\mathrm{T}}\mathbf{X})] = \exp\left\{-\left|\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\vartheta_j\vartheta_k R_G(t_j,t_k)\right|^{\alpha/2}\right\} \tag{7}$$

where $\mathbf{X}^{\mathrm{T}} = [X(t_1), X(t_2), \ldots, X(t_n)]$, $\boldsymbol{\vartheta}^{\mathrm{T}} = [\vartheta_1, \vartheta_2, \ldots, \vartheta_n]$, $\mathbf{t}^{\mathrm{T}} = [t_1, t_2, \ldots, t_n]$ and $R_G(t_j,t_k) = E[G(t_j)G(t_k)]$ is the autocorrelation function of the underlying Gaussian process $G(t)$.

In the next sections, the probabilistic characterization of the response of scalar systems subject to α-stable excitations will be addressed. In particular, an effective approach for evaluating the response PDF and CF in the special case of sub-Gaussian input will be presented. The extension to multi-degree-of-freedom (MDOF) systems is reported in the Appendix.

3 α-STABLE LÉVY NOISE EXCITATION

In analogy to the Gaussian white noise $W_0(t)$, given by the formal derivative of the Wiener process $B_0(t)$, an α-stable Lèvy white noise $W_{L_\alpha}(t)$ may be defined as

$$W_{L_\alpha}(t) = \frac{\mathrm{d}L_\alpha(t)}{\mathrm{d}t} \tag{8}$$

where $L_\alpha(t)$ denotes the corresponding α-stable Lèvy motion process. Zero-shift and zero-skewness processes belonging to this class enjoy the following properties: i) start from zero, that is $L_\alpha(0) = 0$, with probability one; ii) feature stationary and independent increments $L_\alpha(t) - L_\alpha(s)$, $t > s$, having the α-stable distribution $S_\alpha\left((t-s)^{1/\alpha}, 0, 0\right)$, so that the CF of the increment $\mathrm{d}L_\alpha(t)$ takes the form

$$\phi_{\mathrm{d}L_\alpha}(\vartheta) = \exp\left(-\mathrm{d}t\,|\vartheta|^\alpha\right). \tag{9}$$

Notice that for $\alpha \to 2$, $\mathrm{d}L_\alpha(t) \to \sqrt{2}\mathrm{d}B_0(t)$, where $\mathrm{d}B_0(t)$ is the increment of the Wiener process.

Let us now consider a first-order system excited by an α-stable Lèvy white noise

$$\begin{aligned} \dot{Y}(t) &= f(Y,t) + W_{L_\alpha}(t); \quad t > 0 \\ Y(0) &= Y_0 \end{aligned} \tag{10}$$

where a dot over a variable denotes time derivative; $f(Y,t)$ is an arbitrary function; and Y_0 is the initial condition, here assumed to be a zero mean random variable independent of the stochastic excitation $W_{L_\alpha}(t)$.

The evolution of the response PDF, $p_Y(y,t)$, is ruled by the so-called Einstein-Smoluchowsky (ES) equation[15,16]

$$\frac{\partial p_Y(y,t)}{\partial t} = -\frac{\partial}{\partial y}\left[f(y,t)\,p_Y(y,t)\right] + \frac{\partial^\alpha p_Y(y,t)}{\partial |y|^\alpha} \tag{11}$$

where the symbol $\partial^{\alpha}(\cdot)/\partial|y|^{\alpha}$ in the diffusion term denotes the Riesz-Weil fractional derivative[24,25].

For a well-behaved function $p_Y(y,t)$, the following relationship holds:

$$\mathcal{F}\left[\frac{\partial^{\alpha} p_Y(y,t)}{\partial|y|^{\alpha}}\right] = -|\vartheta|^{\alpha}\,\phi_Y(\vartheta,t) \tag{12}$$

where $\mathcal{F}[\cdot]$ is the Fourier Transform operator and $\phi_Y(\vartheta,t)$ is the CF of the response, i.e. $\phi_Y(\vartheta,t) = \mathcal{F}[p_Y(y,t)]$. Taking into account Eq. (12), the Fourier Transform of the ES equation (11) yields the following equation for the response CF:

$$\frac{\partial \phi_Y(\vartheta,t)}{\partial t} = \mathrm{i}\,\vartheta E\left[f(Y,t)e^{\mathrm{i}\vartheta Y}\right] - |\vartheta|^{\alpha}\,\phi_Y(\vartheta,t) \tag{13}$$

which is often called spectral ES equation.

As shown by Di Paola and Failla[23], Eq. (13) can also be built by applying the rules of stochastic differential calculus, thus allowing a straightforward generalization to MDOF systems. Exact solutions of the spectral ES equation have been obtained for some scalar systems with polynomial nonlinearities, only in the stationary case[18,19] ($\partial\phi_Y(\vartheta,t)/\partial t = 0$). Though much more tractable mathematically than the ES equation (11), Eq. (13), in general, should be solved numerically with high computational costs. Recently, an approximate solution procedure based on the joint use of wavelet representation and weighted residual method has been proposed[23].

4 α-STABLE SUB-GAUSSIAN EXCITATION

Let us now assume that the scalar system (10) is subject to a special kind of symmetric α-stable process, namely a sub-Gaussian excitation:

$$\begin{aligned} &\dot{Y}(t) = f(Y,t) + A^{1/2}G(t); \quad t > 0 \\ &Y(0) = Y_0 \end{aligned} \tag{14}$$

where, according to the notation introduced in Section 2, $G(t)$ is a zero mean Gaussian process with assigned autocorrelation function $R_G(t_j,t_k) = E[G(t_j)G(t_k)]$; and A denotes an $\alpha/2$-stable random variable totally skewed to the right ($A \sim S_{\alpha/2}((\cos\pi\alpha/4)^{2/\alpha},1,0)$) and independent of $G(t)$.

The above problem is here tackled observing that Eq. (14) may be regarded as the equation of motion of a scalar system driven by a zero mean Gaussian process $G(t)$ with random amplitude $A^{1/2}$. On the other hand, since the process $A^{1/2}G(t)$ is not ergodic, for each realization of the random variable A, say $a > 0$, the input in Eq. (14) is Gaussian and takes the form $a^{1/2}G(t)$. Relying on the previous observations, the probabilistic characterization of the response process $Y(t)$ under the sub-Gaussian input may be pursued through two successive steps. The first step consists in finding the statistical properties of the random process $\hat{Y}(t)$, ruled by the following first-order differential equation:

$$\dot{\hat{Y}}(t) = f(\hat{Y},t) + a^{1/2}G(t); \quad t>0, \ \rho>0$$
$$\hat{Y}(0) = \hat{Y}_0 \tag{15}$$

which is obtained from Eq. (14) replacing the random variable A with its generic realization a. As will be outlined in detail next, for linear systems exact closed-form expressions of the PDF and the CF of $\hat{Y}(t)$ can be easily obtained by classical random vibration theory, while, in general, approximate procedures are required in the non-linear case. Furthermore, it is noted that such functions will depend on the parameter a, that is $p_{\hat{Y}}(y;a,t)$ and $\phi_{\hat{Y}}(\vartheta;a,t)$.

The second step of the present procedure consists in the probabilistic characterization of the response process $Y(t)$ under the sub-Gaussian input $A^{1/2}G(t)$ by using the statistics of the response $\hat{Y}(t)$ to the Gaussian process $a^{1/2}G(t)$, defined in the previous step. For this purpose, it is observed that within the interval $[a, a+\mathrm{d}a]$ the input $a^{1/2}G(t)$ and the associated response process $\hat{Y}(t)$ occur $p_A(a)\mathrm{d}a$ times, being $p_A(a)$ the PDF of the random variable A. It follows that the PDF and the CF of $Y(t)$ may be obtained simply by performing ensemble average of $p_{\hat{Y}}(y;A,t)$ and $\phi_{\hat{Y}}(\vartheta;A,t)$ over the whole set of realizations of A, once the parameter a has been duly replaced by the $\alpha/2$-stable random variable A, that is:

$$p_Y(y;t) = E[p_{\hat{Y}}(y;A,t)] = \int_0^\infty p_A(a)\, p_{\hat{Y}}(y;a,t)\mathrm{d}a;$$
$$\phi_Y(\vartheta;t) = E[\phi_{\hat{Y}}(\vartheta;A,t)] = \int_0^\infty p_A(a)\phi_{\hat{Y}}(\vartheta;a,t)\mathrm{d}a. \tag{16a,b}$$

In a similar way, the joint PDF and CF at two different time instants t_1 and t_2, can be computed as:

$$p_{Y_1Y_2}(y_1,y_2;t_1,t_2) = E[p_{\hat{Y}_1\hat{Y}_2}(y_1,y_2;A,t_1,t_2)] = \int_0^\infty p_A(a)\, p_{\hat{Y}_1\hat{Y}_2}(y_1,y_2;a,t_1,t_2)\mathrm{d}a;$$
$$\phi_{Y_1Y_2}(\vartheta_1,\vartheta_2;t_1,t_2) = E[\phi_{\hat{Y}_1\hat{Y}_2}(\vartheta_1,\vartheta_2;A,t_1,t_2)] = \int_0^\infty p_A(a)\phi_{\hat{Y}_1\hat{Y}_2}(\vartheta_1,\vartheta_2;a,t_1,t_2)\mathrm{d}a. \tag{17a,b}$$

As regards the second step of the above described procedure, it is worth noting that the evaluation of the integrals in Eq. (16) (or Eq. (17)) may be quite involved. In fact, the PDF of the $\alpha/2$-stable random variable A, $p_A(a)$, which is obtained making the Inverse Fourier Transform of the CF $\phi_A(\vartheta)$ (Eq. (2) where $\sigma = (\cos(\pi\alpha/4))^{2/\alpha}$, $\beta = 1$ and $\mu = 0$), does not take a simple analytical form for any value of α. Since $p_Y(y;t)$ and $\phi_Y(\vartheta;t)$ are the stochastic averages of $p_{\hat{Y}}(y;A,t)$ and $\phi_{\hat{Y}}(\vartheta;A,t)$, respectively, an efficient way to avoid the evaluation of the integrals in Eq. (16) consists in simulating a large number of samples[14] of A, say N, and then applying the following relationships, according to Monte Carlo simulation (MCS) method:

$$p_Y(y;t) = E[p_{\hat{Y}}(y;A,t)] \cong \frac{1}{N}\sum_{j=1}^{N} p_{\hat{Y}}(y;a^{(j)},t);$$
$$\phi_Y(\vartheta;t) = E[\phi_{\hat{Y}}(\vartheta;A,t)] \cong \frac{1}{N}\sum_{j=1}^{N} \phi_{\hat{Y}}(\vartheta;a^{(j)},t) \tag{18a,b}$$

being $a^{(j)}$ the $j-$th realization of A. Obviously, an analogous procedure may be followed to obtain the joint PDF and CF given in Eq. (17).

It is noted that the presented procedure allows to reduce the analysis of a system under sub-Gaussian input to the one of the same system subject to the underlying Gaussian process. In this respect, the probabilistic characterization of the response to α-stable Lèvy white noise, discussed in Section 3, turns out to be much more difficult since it requires the solution of the ES equation or of its spectral counterpart.

In the sequel, the application of the proposed approach will be described in detail considering separately linear and non-linear systems.

4.1 First-order linear systems under sub-Gaussian input

The simple case of a linear half oscillator under sub-Gaussian input is first treated:

$$\begin{aligned}&\dot{Y}(t) = -\rho Y(t) + A^{1/2}G(t); \quad t > 0,\ \rho > 0\\ &Y(0) = Y_0.\end{aligned} \tag{19}$$

The statistics of the response process $Y(t)$ may be easily evaluated taking into account that, since the equation of motion (19) is linear, the following relationship holds:

$$Y(t) = A^{1/2}\tilde{Y}(t) \tag{20}$$

where $\tilde{Y}(t)$ denotes the response of the linear system subject to the underlying Gaussian input $G(t)$, say for $a=1$ ($\dot{\tilde{Y}}(t) = -\rho\tilde{Y}(t) + G(t)$). If the initial condition $\tilde{Y}_0$ is supposed to be a zero mean Gaussian random variable, then $\tilde{Y}(t)$ is a zero mean normal random process. Therefore, it clearly appears that the response $Y(t)$ defined in Eq. (20) is sub-Gaussian $S_\alpha S$ with underlying Gaussian process $\tilde{Y}(t)$. According to Eqs. (4) and (7), the unconditional and joint CFs of $Y(t)$ are given, respectively, by:

$$\phi_Y(\vartheta;t) = \exp\left\{-\left|\frac{1}{2}\vartheta^2\sigma^2_{\tilde{Y}}(t)\right|^{\alpha/2}\right\}; \tag{21}$$

$$\phi_{Y_1Y_2}(\vartheta_1,\vartheta_2;t_1,t_2) = \exp\left\{-\left|\frac{1}{2}\boldsymbol{\vartheta}^{\mathrm{T}}\mathbf{R}_{\tilde{Y}}(t_1,t_2)\boldsymbol{\vartheta}\right|^{\alpha/2}\right\}. \tag{22}$$

Alternatively, the same result may be obtained by applying the proposed procedure. For this purpose, first the statistical properties of the response process $\hat{Y}(t)$ to the Gaussian excitation $a^{1/2}G(t)$ (Eq. (15)) have to be evaluated. For simplicity's sake, the initial condition $\hat{Y}_0$ is supposed to be a zero mean Gaussian random variable, so that the response $\hat{Y}(t)$ is a zero mean Gaussian process, whose complete probabilistic characterization is ensured by the knowledge of the autocorrelation function $R_{\hat{Y}}(t_j,t_k)$. In view of the linearity of the system,

$R_{\hat{Y}}(t_j, t_k)$ may be simply obtained by first evaluating the autocorrelation function $R_{\tilde{Y}}(t_j, t_k)$ of the response process $\tilde{Y}(t)$ to the underlying Gaussian input $G(t)$, say for $a = 1$, and then applying the following relationship:

$$R_{\hat{Y}}(t_j, t_k) = aR_{\tilde{Y}}(t_j, t_k). \tag{23}$$

The procedure for deriving the autocorrelation function of the random process $\tilde{Y}(t)$ can be found in classical textbooks on random vibration theory[2].

According to Eq. (23), once the autocorrelation function $R_{\tilde{Y}}(t_j, t_k)$ is known, the unconditional PDF and CF of the Gaussian process $\hat{Y}(t)$ can be obtained, respectively, as follows:

$$\begin{aligned} p_{\hat{Y}}(y; a, t) &= \frac{1}{\sqrt{2\pi} a^{1/2} \sigma_{\tilde{Y}}(t)} \exp\left(-\frac{y^2}{2a\sigma_{\tilde{Y}}^2(t)} \right); \\ \phi_{\hat{Y}}(\vartheta; a, t) &= \exp\left(-\frac{1}{2} \vartheta^2 a \sigma_{\tilde{Y}}^2(t) \right), \end{aligned} \tag{24a,b}$$

being $\sigma_{\hat{Y}}^2(t) = a\sigma_{\tilde{Y}}^2(t)$ and $\sigma_{\tilde{Y}}^2(t) = R_{\tilde{Y}}(t, t)$. Second-order statistics may be evaluated in terms of joint PDF or CF at two different time instants t_1 and t_2 given, respectively, by:

$$\begin{aligned} p_{\hat{Y}_1\hat{Y}_2}(y_1, y_2; a, t_1, t_2) &= \frac{1}{2\pi a \left(\mathrm{Det}\left(\mathbf{R}_{\tilde{\mathbf{Y}}}(t_1, t_2) \right) \right)^{1/2}} \exp\left\{ -\frac{a^{-1}}{2} \mathbf{y}^{\mathrm{T}} \mathbf{R}_{\tilde{\mathbf{Y}}}^{-1}(t_1, t_2) \mathbf{y} \right\}; \\ \phi_{\hat{Y}_1\hat{Y}_2}(\vartheta_1, \vartheta_2; a, t_1, t_2) &= \exp\left\{ -\frac{a}{2} \boldsymbol{\vartheta}^{\mathrm{T}} \mathbf{R}_{\tilde{\mathbf{Y}}}(t_1, t_2) \boldsymbol{\vartheta} \right\}, \end{aligned} \tag{25a,b}$$

where $\hat{Y}_i = \hat{Y}(t_i)$, $y_i = y(t_i)$, $(i = 1, 2)$, $\mathbf{y}^{\mathrm{T}} = [y_1, y_2]$ and $\boldsymbol{\vartheta}^{\mathrm{T}} = [\vartheta_1, \vartheta_2]$. In Eq. (25), $\mathbf{R}_{\tilde{\mathbf{Y}}}(t_1, t_2)$ denotes the autocorrelation matrix of $\tilde{\mathbf{Y}}^{\mathrm{T}} = \left[\tilde{Y}_1, \tilde{Y}_2 \right]$, ($\tilde{Y}_i = \tilde{Y}(t_i)$, $i = 1, 2$), given by:

$$\mathbf{R}_{\tilde{\mathbf{Y}}}(t_1, t_2) = E[\tilde{\mathbf{Y}}\tilde{\mathbf{Y}}^{\mathrm{T}}] = \begin{bmatrix} \sigma_{\tilde{\mathbf{Y}}}^2(t_1) & R_{\tilde{\mathbf{Y}}}(t_1, t_2) \\ R_{\tilde{\mathbf{Y}}}(t_2, t_1) & \sigma_{\tilde{\mathbf{Y}}}^2(t_2) \end{bmatrix}. \tag{26}$$

Notice that both the PDF and the CF of $\hat{Y}(t)$ may be regarded as functions of a random parameter since they depend on the generic realization of the random variable A.

Then, following the proposed approach, the unconditional (or joint) PDF and CF of $Y(t)$ may be obtained simply by evaluating the integrals in Eq. (16) (or Eq. (17)).

Summarizing, the presented procedure for the probabilistic characterization of the response of a linear system driven by a sub-Gaussian process $A^{1/2}G(t)$ requires:

i) to compute the autocorrelation function $R_{\hat{Y}}(t_j, t_k)$ of the response $\hat{Y}(t)$ to the Gaussian process $a^{1/2}G(t)$ through Eq. (23);

ii) to evaluate the unconditional (or joint) PDF and CF of the Gaussian response process $\hat{Y}(t)$ by means of Eq. (24) (or Eq. (25));

iii) to perform ensemble average of $p_{\hat{Y}}(y;A,t)$ and $\phi_{\hat{Y}}(\vartheta;A,t)$ (or $p_{\hat{Y}_1\hat{Y}_2}(y_1,y_2;A,t_1,t_2)$ and $\phi_{\hat{Y}_1\hat{Y}_2}(\vartheta_1,\vartheta_2;A,t_1,t_2)$) according to Eq. (16) (or Eq. (17)).

4.2 First-order non-linear systems under sub-Gaussian input

Let us now consider the general case in which $f(Y,t)$ in Eq. (14) is an arbitrary non-linear function.

Since the system is non-linear, Eq. (20) does not apply and, in general, closed-form solutions in terms of PDF or CF cannot be derived. The proposed approach first requires to evaluate the statistics of the random process $\hat{Y}(t)$ ruled by Eq. (15). However, due to the nonlinearity of the system, the response $\hat{Y}(t)$ to the zero mean Gaussian process $a^{1/2}G(t)$ is non-Gaussian and the evaluation of its exact PDF or CF is a very hard task. Nevertheless, in the case in which $G(t)$ is a normal white noise, analytical expressions of the response PDF are available for some special classes of non-linear systems[5,26-28]. Once the exact or approximate PDF and CF of $\hat{Y}(t)$ are known, the probabilistic characterization of the response process $Y(t)$ under the sub-Gaussian input can be still performed by applying Eqs. (16) and (17). Indeed, such equations are not related to the Gaussianity of the process $\hat{Y}(t)$ since they stem from the interpretation of the sub-Gaussian input as a conditional Gaussian process.

In order to clarify the concepts stated above, let us assume that the underlying Gaussian process $G(t)$ is a zero mean normal white noise, i.e. $G(t)\equiv W_0(t)$, fully characterized by the autocorrelation function:

$$R_{W_0}(t_1,t_2)=E[W_0(t_1)W_0(t_2)]=q\delta(t_1-t_2) \tag{27}$$

where $\delta(\cdot)$ denotes the Dirac's delta function and $q=2\pi S_0$, being S_0 the Power Spectral Density (PSD) of $W_0(t)$. In this case, the input in Eq. (14), $A^{1/2}G(t)\equiv A^{1/2}W_0(t)$, will be termed *sub-Gaussian white noise* or *subordinate α-stable white noise* and is here denoted as $W_\alpha(t)$. It is recalled that the white noise $W_0(t)$ is the formal derivative of the Wiener process $B_0(t)$, that is $W_0(t)=\mathrm{d}B_0(t)/\mathrm{d}t$ ($E[B_0(t)]=0$; $E[B_0^2(t)]=q\mathrm{d}t$). Similarly, the sub-Gaussian white noise $W_\alpha(t)$ may be defined as the formal derivative of the *sub-Gaussian Wiener process*, $B_\alpha(t)=A^{1/2}B_0(t)$. The process $B_\alpha(t)$ enjoys some important properties:

i) it has independent stationary increments ($B_\alpha(t)-B_\alpha(s)$, $t>s$) following the α-stable distribution, that is $B_\alpha(t)-B_\alpha(s)\sim S_\alpha(((t-s)/2)^{1/2},0,0)$, $t>0$;

ii) the CF of an increment of the sub-Gaussian Wiener process, $\mathrm{d}B_\alpha(t)=A^{1/2}\mathrm{d}B_0(t)$, takes the form:

$$\phi_{\mathrm{d}B_\alpha(t)}(\vartheta)=\exp\left(-(\mathrm{d}t/2)^{\alpha/2}\left|\vartheta\right|^{\alpha}\right); \tag{28}$$

iii) for $\alpha\to 2$, $\mathrm{d}B_\alpha(t)\to\mathrm{d}B_0(t)$, so that the Wiener process may be viewed as a particular case of the process $B_\alpha(t)$.

By virtue of property i), the main tools of the Itô stochastic differential calculus can be used for analyzing the response of non-linear systems driven by sub-Gaussian white noises. In particular, since the increments of the subordinate Wiener process $B_\alpha(t)$ are independent, one may take full advantage of the non-anticipating property.

A comparison may be reasonably made between the sub-Gaussian white noise $W_\alpha(t)$ and the Lévy white noise $W_{L_\alpha}(t)$ (see Section 2) as both processes are α-stable. Specifically, it can be observed that property i) is fulfilled for both the increments $\mathrm{d}B_\alpha(t) = A^{1/2}\mathrm{d}B_0(t)$ and $\mathrm{d}L_\alpha(t) = W_{L_\alpha}(t)\mathrm{d}t$, whereas the CFs $\phi_{\mathrm{d}B_\alpha(t)}(\vartheta)$ (Eq. (28)) and $\phi_{\mathrm{d}L_\alpha(t)}(\vartheta)$ (Eq. (9)), are quite different. By comparing Eqs. (28) and (9), it can be inferred that the processes $W_\alpha(t)$ and $W_{L_\alpha}(t)$ have different scales.

Once the basic features of the input process $W_\alpha(t) = A^{1/2}W_0(t)$ have been outlined, Eq. (14) (where $G(t) \equiv W_0(t)$) may be converted into the standard Itô form as follows:

$$\mathrm{d}Y(t) = f(Y,t)\mathrm{d}t + A^{1/2}\mathrm{d}B_0(t). \tag{29}$$

Then, replacing the random variable A with its generic realization a, the Itô type equation for the response process $\hat{Y}(t)$ to the input $a^{1/2}W_0(t)$ is obtained

$$\mathrm{d}\hat{Y}(t) = f(\hat{Y},t)\mathrm{d}t + a^{1/2}\mathrm{d}B_0(t). \tag{30}$$

Notice that in Eq. (29) the input is represented by an increment of the sub-Gaussian Wiener process $\mathrm{d}B_\alpha(t) = A^{1/2}\mathrm{d}B_0(t)$, while in Eq. (30) the system is subject to an increment of the Wiener process $\mathrm{d}B_0(t)$ multiplied by the square root of the generic realization of the random variable A. As already mentioned, the first step of the proposed procedure consists in evaluating the statistics of $\hat{Y}(t)$. Such process is a Markov one whose unconditional PDF $p_{\hat{Y}}(y;a,t)$ is ruled by the well-known Fokker-Planck-Kolmogorov (FPK) equation:

$$\frac{\partial p_{\hat{Y}}(y;a,t)}{\partial t} = -\frac{\partial}{\partial y}\left(f(y,t)\,p_{\hat{Y}}(y;a,t)\right) + \frac{1}{2}aq\frac{\partial^2 p_{\hat{Y}}(y;a,t)}{\partial y^2}. \tag{31}$$

Alternatively, $\hat{Y}(t)$ may be characterized evaluating the CF, $\phi_{\hat{Y}}(\vartheta;a,t)$, as solution of the following partial differential equation:

$$\frac{\partial \phi_{\hat{Y}}(\vartheta;a,t)}{\partial t} = \mathrm{i}\,\vartheta E[f(\hat{Y},t)\exp(\mathrm{i}\vartheta\hat{Y})] - \frac{1}{2}aq\vartheta^2\phi_{\hat{Y}}(\vartheta;a,t) \tag{32}$$

which can be obtained simply by making the Fourier Transform of the FPK equation (31). If $f(\hat{Y},t)$ is given by a polynomial of the type $f(\hat{Y},t) = \sum_{k=1}^{m} c_k \hat{Y}^k(t)$, taking into account the well-known relationship:

$$E[\hat{Y}^k \exp(\mathrm{i}\vartheta\hat{Y})] = (-\mathrm{i})^k \frac{\partial^k \phi_{\hat{Y}}(\vartheta;a,t)}{\partial \vartheta^k}, \tag{33}$$

the equation ruling the CF takes the form:

$$\frac{\partial \phi_{\hat{Y}}(\vartheta;a,t)}{\partial t} = \mathrm{i}\,\vartheta\sum_{k=1}^{m} c_k(-\mathrm{i})^k \frac{\partial^k \phi_{\hat{Y}}(\vartheta;a,t)}{\partial \vartheta^k} - \frac{1}{2}aq\vartheta^2\phi_{\hat{Y}}(\vartheta;a,t). \tag{34}$$

Both Eqs. (31) and (32) (or (34)) should be supplemented by the appropriate boundary and initial conditions.

Unfortunately, the analytical solution of the previous partial differential equations presents severe difficulties, so that several approximate techniques have been developed in the literature. Nevertheless, closed-form stationary solutions of the FPK equation (31) can be obtained for some special non-linear systems belonging to the class of *generalized stationary potential*[5,26-28]. For such systems the proposed approach still represents an accurate and efficient tool since the stationary PDF of the response under sub-Gaussian input can be easily obtained by means of Eq. (16a), once the exact stationary PDF of the response to the underlying Gaussian process multiplied by $a^{1/2}$ is known.

5 NUMERICAL APPLICATIONS

5.1 Stationary response of a linear half oscillator under sub-Gaussian input

As first example let us consider the one-dimensional linear system under sub-Gaussian input ruled by Eq. (19), here rewritten for clarity's sake:

$$\dot{Y}(t) = -\rho Y(t) + A^{1/2} G(t);\ \rho > 0. \tag{35}$$

Let $G(t)$ be a zero mean stationary Gaussian process fully characterized by the following autocorrelation function, $R_G(\tau)$:

$$R_G(\tau) = \sigma_G^2 \exp\left(-\nu|\tau|\right);\ \nu > 0. \tag{36}$$

If the motion starts at $t = -\infty$, then the response process $Y(t)$ is stationary too. Furthermore, as outlined in Section 4.1, $Y(t)$ is an α-stable sub-Gaussian process whose CF, $\phi_Y(\vartheta)$, can be determined analytically through Eq. (21), once the autocorrelation function, $R_{\tilde{Y}}(\tau)$, of the response $\tilde{Y}(t)$ to the underlying Gaussian process $G(t)$ is known. The random process $\tilde{Y}(t)$, solution of the differential equation $\dot{\tilde{Y}}(t) = -\rho\tilde{Y}(t) + G(t)$, is a zero mean stationary Gaussian one whose autocorrelation function $R_{\tilde{Y}}(\tau)$ takes the following form:

$$R_{\tilde{Y}}(\tau) = \frac{\sigma_G^2}{\rho\left(\rho^2 - \nu^2\right)}\left\{\rho\cosh(\nu\tau) - \nu\cosh(\rho\tau) + \mathrm{sgn}(\tau)\left[\nu\sinh(\rho\tau) - \rho\sinh(\nu\tau)\right]\right\}. \tag{37}$$

Then, the exact CF of the stationary response $Y(t)$, $\phi_Y(\vartheta)$, is obtained from Eq. (21) setting $\sigma_{\tilde{Y}}^2(t) = \sigma_{\tilde{Y}}^2$, where $\sigma_{\tilde{Y}}^2 = R_{\tilde{Y}}(0) = \sigma_G^2 / \rho(\rho+\nu)$. The exact stationary PDF, $p_Y(y)$, may be evaluated making the Inverse Fourier Transform of the CF $\phi_Y(\vartheta)$. Similarly, second-order statistics can be deduced in terms of the exact joint CF $\phi_{Y_1Y_2}(\vartheta_1, \vartheta_2; \tau)$ by means of Eq. (22).

In order to apply the procedure presented in the paper, first the statistics of $\hat{Y}(t)$, response to the Gaussian input $a^{1/2}G(t)$, need to be calculated by using Eqs. (24) and (25). For this purpose, the autocorrelation function $R_{\hat{Y}}(\tau)$ can be evaluated substituting $R_{\tilde{Y}}(\tau)$ as given in Eq. (37) into Eq. (23). Then, the unconditional and joint PDF and CF of the response $Y(t)$ to the sub-Gaussian input can be obtained performing ensemble averages according to Eqs. (16) and (17).

The analysis has been carried out for different values of the stability index α, selecting the parameters in Eqs. (35) and (36) as follows: $\rho = 0.6$, $\sigma_G^2 = 1$ and $\nu = 0.8$. In Fig. 1, the stationary CF of $Y(t)$, $\phi_Y(\vartheta)$, evaluated by means of the proposed approach is compared with the exact one (Eq. (21)). An analogous comparison in terms of stationary PDF, $p_Y(y)$, is shown in Fig. 2, where the exact solution is now derived making the Inverse Fourier Transform of the CF (21). The stochastic averages in Eq. (16) have been computed both evaluating numerically the integrals and by MCS, namely generating $N = 10000$ samples of the random variable A and then applying Eq. (18).

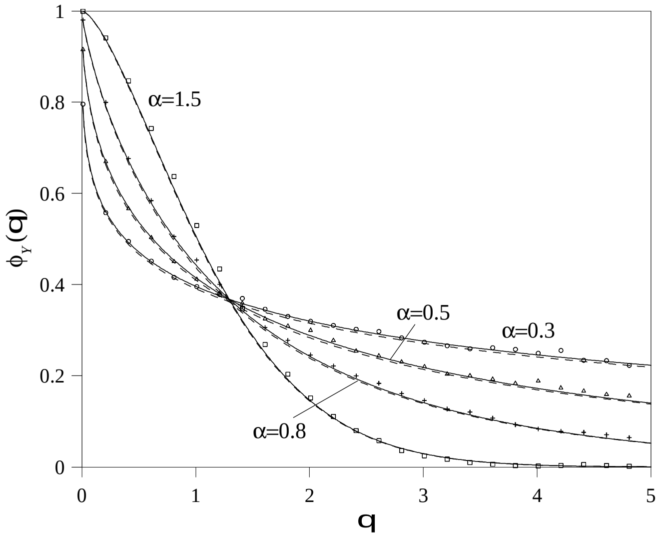

Figure 1: Stationary CF of the response of the linear half oscillator in Eq. (35) for different values of α: exact solution (Eq. (21)) coincident with the proposed one (Eq. (16b)), (continuous line); proposed solution computed by MCS (Eq. (18b)), (dashed line); classical MCS (symbols).

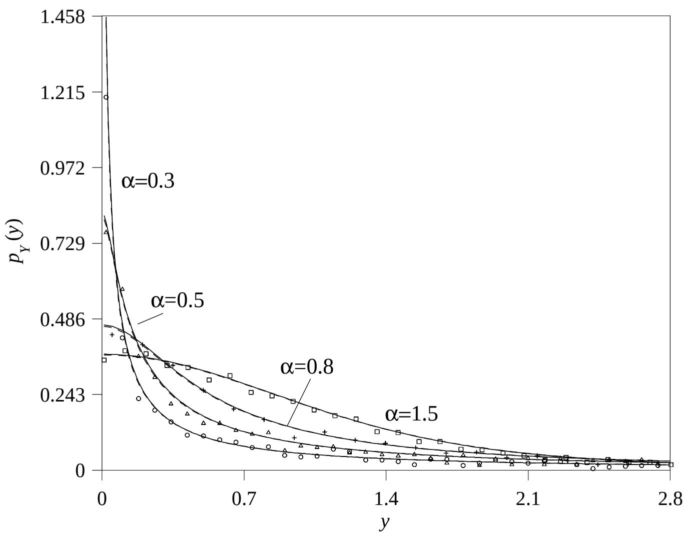

Figure 2: Stationary PDF of the response of the linear half oscillator in Eq. (35) for different values of α: exact solution (Inverse Fourier Transform of Eq. (21)) coincident with the proposed one (Eq. (16a)), (continuous line); proposed solution computed by MCS (Eq. (18a)), (dashed line); classical MCS (symbols).

As shown in Figs. 1 and 2, the results obtained through the numerical evaluation of the integrals in Eq. (16) obviously coincide with the exact solutions, but to retrieve the PDF $p_A(a)$ as Inverse Fourier Transform of the characteristic function $\phi_A(\vartheta)$ has not been an easy task due to the heavy tailed distribution of the $\alpha/2$-stable random variable A. Conversely, the application of digital simulation according to Eq. (18) is more straightforward and robust. At last, in Figs. 1 and 2 the results deduced by applying brute classical MCS to Eq. (35) are also plotted. Notice that the proposed procedure yields accurate estimates in terms of both PDF and CF of the response even for small values of the characteristic exponent α.

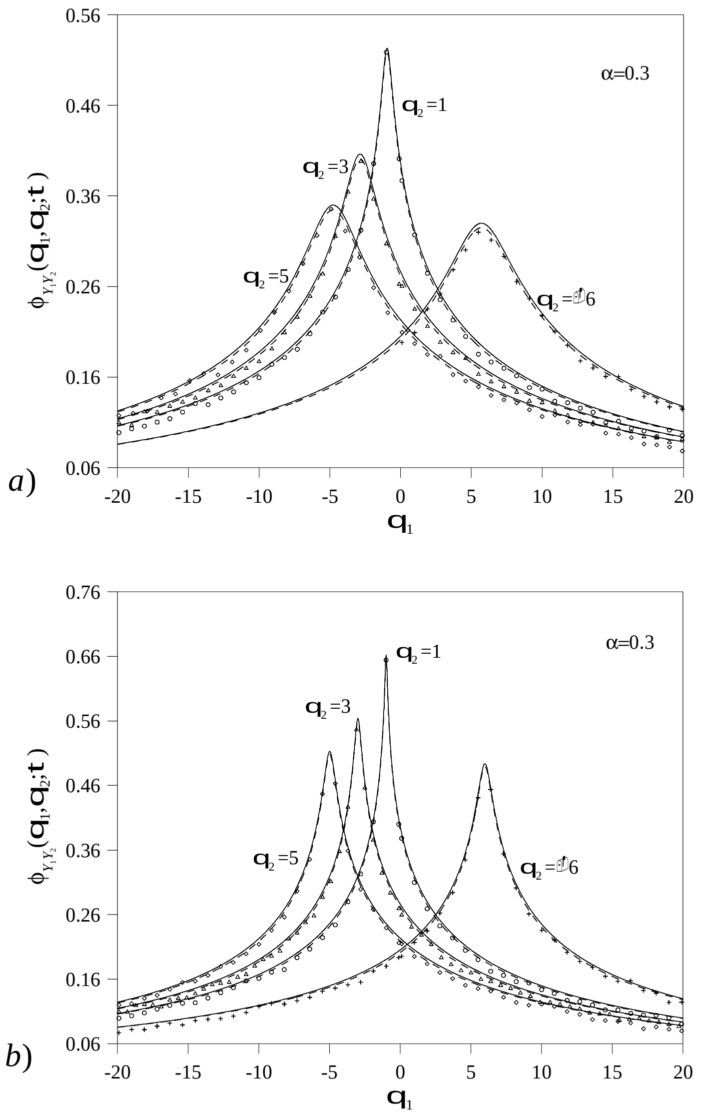

Figure 3: Stationary joint CF of the response of the linear half oscillator in Eq. (35) for different values of ϑ_2, (a) $\tau = 0.1$ s and (b) $\tau = 0.5$ s: exact solution (Eq. (22)) coincident with the proposed one (Eq. (17b)), (continuous line); proposed solution computed by MCS (dashed line); classical MCS (symbols).

Figure 3 displays the stationary joint CF of $Y(t)$, $\phi_{Y_1Y_2}(\vartheta_1, \vartheta_2; \tau)$, for various values of ϑ_2. The stability index α is set equal to 0.3 and two different choices of $\tau = t_2 - t_1$, say $\tau = 0.1$ s and $\tau = 0.5$ s, are considered. The exact solution given by Eq. (22) is compared with the proposed one (Eq. (17b)) as well as with the results provided by classical MCS. Specifically, the stochastic average in Eq. (17b) has been computed both solving numerically the integral and by MCS (with $N = 10000$ samples) following the procedure outlined above for the unconditional PDF and CF. It can be observed that the results obtained by the present approach are in good agreement with the exact solution and classical MCS data. In particular, when the integral in Eq. (17b) is computed numerically, by preliminarily evaluating the PDF of the random variable A, as expected, the proposed solution coincides with the exact one. Further numerical investigations, here omitted for conciseness, have demonstrated that varying the stability index α in the range $(0,2]$, Eq. (17b) still provides accurate estimates of the joint CF of $Y(t)$.

The results discussed above state that the probabilistic characterization of the response of linear systems under sub-Gaussian input may be pursued following four different ways: 1) exact solution (Eqs. (21), (22) and corresponding Inverse Fourier Transforms to obtain the unconditional and joint PDF, respectively); 2) proposed method based on the numerical evaluation of the integrals in Eqs. (16) and (17); 3) proposed approach associated with MCS (see Eq. (18)); 4) classical MCS.

5.2 Stationary response of a non-linear half oscillator driven by a sub-Gaussian white noise

The second example concerns the probabilistic characterization of the response of the following non-linear system:

$$\dot{Y}(t) = -\rho Y(t) - \eta Y^3(t) + A^{1/2} W_0(t);\ \rho > 0, \eta > 0 \tag{38}$$

where $W_0(t)$ is a zero mean stationary Gaussian white noise with autocorrelation function $R_{W_0}(\tau) = E[W_0(t)W_0(t+\tau)] = q\delta(\tau)$. As pointed out in Section 4.2, since the system is non-linear, Eq. (20) does not apply, whereas the relationships in Eqs. (16) and (17) still hold. Let us then consider the non-linear half oscillator in Eq. (38) subject to the Gaussian input $a^{1/2}W_0(t)$:

$$\dot{\hat{Y}}(t) = -\rho \hat{Y}(t) - \eta \hat{Y}^3(t) + a^{1/2} W_0(t) \tag{39}$$

being $a > 0$ the generic realization of the random variable A. The above system belongs to the class of *generalized stationary potential*[5]. In particular, the stationary PDF of the non-Gaussian response process $\hat{Y}(t)$ is known to be:

$$p_{\hat{Y}}(y;a) = C(a)\exp\left[-\frac{2}{aq}\left(\rho\frac{y^2}{2} + \eta\frac{y^4}{4}\right)\right] \tag{40}$$

where $C(a)$ is a function of the parameter a such that $p_{\hat{Y}}(y;a)$ satisfies the normalization condition. Then, as stated by Eq. (16a), the stationary PDF of the response $Y(t)$ to the sub-Gaussian white noise, $p_Y(y)$, can be obtained performing ensemble average of $p_{\hat{Y}}(y;A)$.

Figure 4 shows the stationary response PDF, $p_Y(y)$, evaluated for different values of the stability index α and the following selection of the parameters: $\rho = 0.5$, $\eta = 0.3$ and $q = 1$. The results obtained by applying the proposed procedure are contrasted with those provided by classical MCS. As in the previous example, the stochastic average, defining the stationary

PDF $p_Y(y)$ of the response, has been computed both evaluating numerically the integral in Eq. (16a) and by MCS generating $N = 10000$ samples of the random variable A (Eq. (18a)). Notice that also in the non-linear case the proposed procedure yields very accurate estimates of the response statistics for different values of the characteristic exponent α.

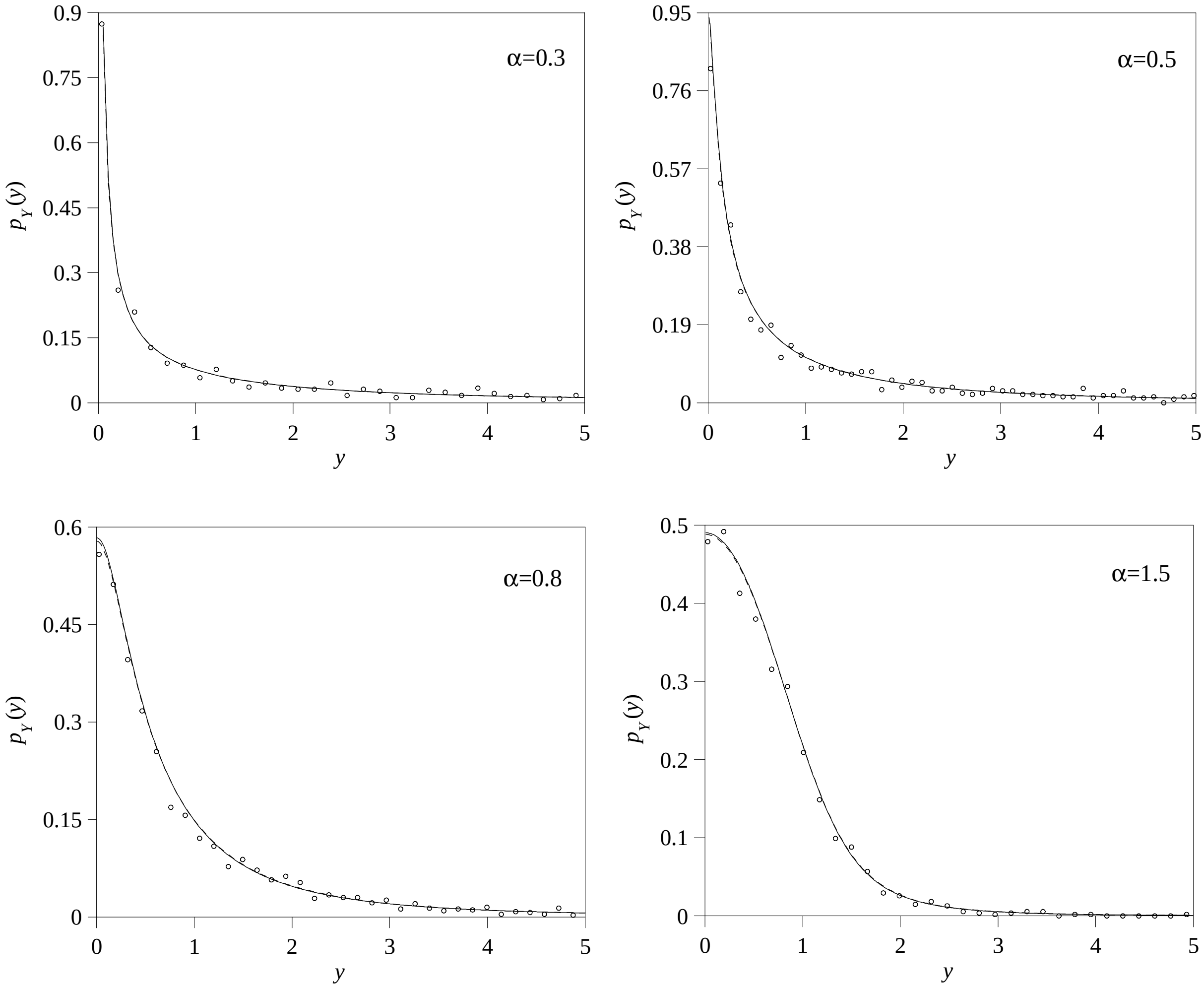

Figure 4: Stationary PDF of the response of the non-linear half oscillator in Eq. (38) for different values of α: proposed solution (Eq. (16a)), (continuous line); proposed solution computed by MCS (Eq. (18a)), (dashed line); classical MCS (symbols).

It has to be emphasized that classical MCS is much more onerous than the present procedure even in the case in which the stochastic averages in Eqs. (16) and (17) are computed resorting to digital simulation. In fact, the application of classical MCS involves the following steps: i) simulate a sample, $a^{(j)}$, of the random variable $A \sim S_{\alpha/2}((\cos(\pi\alpha/4))^{2/\alpha},1,0)$; ii) generate a sample, $W_0^{(j)}(t)$, of the stationary white noise $W_0(t)$, for instance by means of the wave superposition-based technique proposed by Shinozuka[29]; iii) multiply $a^{(j)}$ by $W_0^{(j)}(t)$ to obtain the j-th sample, $a^{(j)}W_0^{(j)}(t)$, of the sub-Gaussian white noise; iv) evaluate the response $Y^{(j)}(t)$ to the sample function $a^{(j)}W_0^{(j)}(t)$ by integrating the equation of motion; v) repeat the procedure for a large number of samples; vi) evaluate the desired statistics of the response by processing the samples $Y^{(j)}(t)$. It follows that in the non-linear case, since no exact solutions are available, the most efficient way to perform the probabilistic characterization of the response to sub-Gaussian input, among those herein examined, consists in the joint application of the proposed procedure and MCS according to Eq. (18).

6 CONCLUSIONS

A method for evaluating the probability density function and characteristic function of the response of linear and non-linear systems driven by α-stable sub-Gaussian processes has been presented. The main idea is that the sub-Gaussian input may be viewed as a conditional Gaussian process, namely as a Gaussian process (the underlying one) having random amplitude (the square root of an $\alpha/2$-stable random variable). So operating, the statistics of the system response to the sub-Gaussian input can be obtained from those of the response to the conditional Gaussian process simply performing ensemble averages with respect to the random amplitude. It has also been shown that in the linear case the characteristic function can be determined in closed-form since the response process is a sub-Gaussian one. According to the present procedure, the probabilistic characterization of the response to α-stable sub-Gaussian input actually exhibits the same difficulties as in the case in which the relevant system is driven by a Gaussian process. It follows that the main tools of classical random vibration theory can be still exploited when the input process is a sub-Gaussian one. In particular, if the underlying Gaussian process is a white noise, one may take full advantage of the Itô stochastic differential calculus.

The accuracy of the proposed approach has been assessed through numerical applications concerning both linear and non-linear one-dimensional systems under sub-Gaussian input. In the linear case the estimates of the response statistics have been shown to be in good agreement with the exact solutions. On the other hand, appropriate comparisons with Monte Carlo simulation results have demonstrated the effectiveness of the present procedure even when system nonlinearities are involved.

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APPENDIX – Multi-degree-of-freedom (MDOF) systems under sub-Gaussian input

In this Appendix, the formulation presented in the paper for one-dimensional systems is properly extended to the case of multi-degree-of-freedom (MDOF) systems.

Let the equations of motion of a n-DOF system driven by a sub-Gaussian input be given in the following form:

$$\begin{aligned}&\dot{\mathbf{Y}}(t)=\mathbf{f}(\mathbf{Y}(t),t)+A^{1/2}\mathbf{G}(t);\\&\mathbf{Y}(0)=\mathbf{Y}_0,\end{aligned}\tag{A.1}$$

where $\mathbf{Y}(t)=\left[Y_1(t),\ Y_2(t),\ \ldots,\ Y_n(t)\right]^{\mathrm{T}}$; $\mathbf{f}(\mathbf{Y}(t),t)$ is a n-vector listing arbitrary linear or non-linear functions of $\mathbf{Y}(t)$ and t; $\mathbf{G}(t)$ is a vector of order n collecting zero mean Gaussian processes with assigned autocorrelation matrix; A denotes an $\alpha/2$-stable random variable totally skewed to the right ($A \sim S_{\alpha/2}((\cos\pi\alpha/4)^{2/\alpha},1,0)$) and independent of $\mathbf{G}(t)$; $\mathbf{Y}_0$ is the n-vector of initial conditions, here supposed to be a zero mean random vector with given covariance matrix, independent of $\mathbf{G}(t)$.

In analogy with the one-dimensional case, the probabilistic characterization of the vector process $\mathbf{Y}(t)$ is performed through two successive steps. The first step consists in finding the response statistics of the following system:

$$\begin{aligned}&\dot{\hat{\mathbf{Y}}}(t)=\mathbf{f}(\hat{\mathbf{Y}}(t),t)+a^{1/2}\mathbf{G}(t);\\&\hat{\mathbf{Y}}(0)=\hat{\mathbf{Y}}_0,\end{aligned}\tag{A.2}$$

where $a>0$ is a real parameter representing the generic realization of the random variable A and $\hat{\mathbf{Y}}_0$ is supposed to be a zero mean Gaussian random vector with given covariance matrix.

In the case in which $\mathbf{f}(\hat{\mathbf{Y}}(t),t)$ is a vector collecting linear functions, the response vector $\hat{\mathbf{Y}}(t)$ is a zero mean Gaussian one whose unconditional PDF and CF, $p_{\hat{\mathbf{Y}}}(\mathbf{y};a,t)$ and $\phi_{\hat{\mathbf{Y}}}(\boldsymbol{\vartheta};a,t)$, are known in explicit form as follows:

$$p_{\hat{\mathbf{Y}}}(\mathbf{y};a,t)=\frac{1}{\sqrt{(2\pi)^n}\,(Det(\boldsymbol{\Sigma}_{\hat{\mathbf{Y}}}(a;t)))^{1/2}}\exp\left\{-\frac{1}{2}\mathbf{y}^{\mathrm{T}}\boldsymbol{\Sigma}_{\hat{\mathbf{Y}}}^{-1}(a;t)\mathbf{y}\right\};\tag{A.3}$$

$$\phi_{\hat{\mathbf{Y}}}(\boldsymbol{\vartheta};a,t)=\exp\left\{-\frac{1}{2}\boldsymbol{\vartheta}^{\mathrm{T}}\boldsymbol{\Sigma}_{\hat{\mathbf{Y}}}(a;t)\boldsymbol{\vartheta}\right\},\tag{A.4}$$

being $\mathbf{\Sigma}_{\hat{\mathbf{Y}}}(a;t)=E[\hat{\mathbf{Y}}(t)\hat{\mathbf{Y}}^{\mathrm{T}}(t)]$ the covariance matrix of $\hat{\mathbf{Y}}(t)$. The joint PDF and CF of $\hat{\mathbf{Y}}(t)$ at two different time instants t_1 and t_2 are given, respectively, by:

$$p_{\hat{\mathbf{Y}}_1\hat{\mathbf{Y}}_2}(\mathbf{y}_1,\mathbf{y}_2;a,t_1,t_2)=\frac{1}{\sqrt{(2\pi)^n}\,(Det(\mathbf{R}_{\hat{\mathbf{Y}}_1\hat{\mathbf{Y}}_2}(a;t_1,t_2)))^{1/2}}\exp\left\{-\frac{1}{2}\begin{pmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{pmatrix}^{\mathrm{T}}\mathbf{R}^{-1}_{\hat{\mathbf{Y}}_1\hat{\mathbf{Y}}_2}(a;t_1,t_2)\begin{pmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{pmatrix}\right\}; \quad (A.5)$$

$$\phi_{\hat{\mathbf{Y}}_1\hat{\mathbf{Y}}_2}(\boldsymbol{\vartheta}_1,\boldsymbol{\vartheta}_2;a,t_1,t_2)=\exp\left\{-\frac{1}{2}\begin{pmatrix}\boldsymbol{\vartheta}_1\\ \boldsymbol{\vartheta}_2\end{pmatrix}^{\mathrm{T}}\mathbf{R}_{\hat{\mathbf{Y}}_1\hat{\mathbf{Y}}_2}(a;t_1,t_2)\begin{pmatrix}\boldsymbol{\vartheta}_1\\ \boldsymbol{\vartheta}_2\end{pmatrix}\right\}, \quad (A.6)$$

where $\hat{\mathbf{Y}}_i=\hat{\mathbf{Y}}(t_i)$, $\mathbf{y}_i=\mathbf{y}(t_i)$, ($i=1,2$), and $\mathbf{R}_{\hat{\mathbf{Y}}_1\hat{\mathbf{Y}}_2}(a;t_1,t_2)$ is defined as:

$$\mathbf{R}_{\hat{\mathbf{Y}}_1\hat{\mathbf{Y}}_2}(a;t_1,t_2)=\begin{bmatrix}E[\hat{\mathbf{Y}}_1\hat{\mathbf{Y}}_1^{\mathrm{T}}] & E[\hat{\mathbf{Y}}_1\hat{\mathbf{Y}}_2^{\mathrm{T}}]\\ E[\hat{\mathbf{Y}}_2\hat{\mathbf{Y}}_1^{\mathrm{T}}] & E[\hat{\mathbf{Y}}_2\hat{\mathbf{Y}}_2^{\mathrm{T}}]\end{bmatrix}. \quad (A.7)$$

In view of the linearity of the system, $\mathbf{\Sigma}_{\hat{\mathbf{Y}}}(a;t)$ and $\mathbf{R}_{\hat{\mathbf{Y}}_1\hat{\mathbf{Y}}_2}(a;t_1,t_2)$ can be evaluated by means of the following relationships:

$$\mathbf{\Sigma}_{\hat{\mathbf{Y}}}(a;t)=a\mathbf{\Sigma}_{\tilde{\mathbf{Y}}}(t);\quad \mathbf{R}_{\hat{\mathbf{Y}}_1\hat{\mathbf{Y}}_2}(a;t_1,t_2)=a\mathbf{R}_{\tilde{\mathbf{Y}}_1\tilde{\mathbf{Y}}_2}(t_1,t_2) \quad (A.8)$$

where $\tilde{\mathbf{Y}}(t)$ denotes the solution of Eq. (A.2) for $a=1$ ($\dot{\tilde{\mathbf{Y}}}(t)=\mathbf{f}(\tilde{\mathbf{Y}}(t),t)+\mathbf{G}(t)$).

Taking into account that the sub-Gaussian input $A^{1/2}\mathbf{G}(t)$ may be viewed as a conditional Gaussian vector process, the unconditional PDF and CF of $\mathbf{Y}(t)$ can be obtained performing ensemble average of $p_{\hat{\mathbf{Y}}}(\mathbf{y};A,t)$ and $\phi_{\hat{\mathbf{Y}}}(\boldsymbol{\vartheta};A,t)$, respectively:

$$\begin{aligned}p_{\mathbf{Y}}(\mathbf{y};t)&=E[p_{\hat{\mathbf{Y}}}(\mathbf{y};A,t)]=\int_0^\infty p_A(a)\,p_{\hat{\mathbf{Y}}}(\mathbf{y};a,t)\mathrm{d}a;\\ \phi_{\mathbf{Y}}(\boldsymbol{\vartheta};t)&=E[\phi_{\hat{\mathbf{Y}}}(\boldsymbol{\vartheta};A,t)]=\int_0^\infty p_A(a)\phi_{\hat{\mathbf{Y}}}(\boldsymbol{\vartheta};a,t)\mathrm{d}a,\end{aligned} \quad (A.9a,b)$$

being $p_A(a)$ the PDF of the random variable A. In a similar way, the joint PDF and CF of $\mathbf{Y}(t)$ at two different time instants t_1 and t_2 can be evaluated as follows:

$$\begin{aligned}p_{\mathbf{Y}_1\mathbf{Y}_2}(\mathbf{y}_1,\mathbf{y}_2;t_1,t_2)&=E[p_{\hat{\mathbf{Y}}_1\hat{\mathbf{Y}}_2}(\mathbf{y}_1,\mathbf{y}_2;A,t_1,t_2)]=\int_0^\infty p_A(a)\,p_{\hat{\mathbf{Y}}_1\hat{\mathbf{Y}}_2}(\mathbf{y}_1,\mathbf{y}_2;a,t_1,t_2)\mathrm{d}a;\\ \phi_{\mathbf{Y}_1\mathbf{Y}_2}(\boldsymbol{\vartheta}_1,\boldsymbol{\vartheta}_2;t_1,t_2)&=E[\phi_{\hat{\mathbf{Y}}_1\hat{\mathbf{Y}}_2}(\boldsymbol{\vartheta}_1,\boldsymbol{\vartheta}_2;A,t_1,t_2)]=\int_0^\infty p_A(a)\phi_{\hat{\mathbf{Y}}_1\hat{\mathbf{Y}}_2}(\boldsymbol{\vartheta}_1,\boldsymbol{\vartheta}_2;a,t_1,t_2)\mathrm{d}a.\end{aligned} \quad (A.10a,b)$$

It has to be mentioned that the probabilistic descriptors above defined can be obtained in closed-form taking into account that since the system is linear, the following relationship holds:

$$\mathbf{Y}(t)=A^{1/2}\tilde{\mathbf{Y}}(t). \quad (A.11)$$

According to Eq. (A.11), the response vector $\mathbf{Y}(t)$ is sub-Gaussian with underlying Gaussian vector process $\tilde{\mathbf{Y}}(t)$, so that its unconditional and joint CFs are given, respectively, by (see Eq. (6)):

$$\phi_{\mathbf{Y}}(\boldsymbol{\vartheta};t)=\exp\left\{-\left|\frac{1}{2}\boldsymbol{\vartheta}^{\mathrm{T}}\boldsymbol{\Sigma}_{\tilde{\mathbf{Y}}}(t)\boldsymbol{\vartheta}\right|^{\alpha/2}\right\}; \tag{A.12}$$

$$\phi_{\mathbf{Y}_1\mathbf{Y}_2}(\boldsymbol{\vartheta}_1,\boldsymbol{\vartheta}_2;t_1,t_2)=\exp\left\{-\left|\frac{1}{2}\begin{pmatrix}\boldsymbol{\vartheta}_1\\ \boldsymbol{\vartheta}_2\end{pmatrix}^{\mathrm{T}}\mathbf{R}_{\tilde{\mathbf{Y}}_1\tilde{\mathbf{Y}}_2}(t_1,t_2)\begin{pmatrix}\boldsymbol{\vartheta}_1\\ \boldsymbol{\vartheta}_2\end{pmatrix}\right|^{\alpha/2}\right\}. \tag{A.13}$$

In the case of MDOF non-linear systems under sub-Gaussian input, the statistics of the response can be still evaluated by means of Eqs. (A.9) and (A.10), while Eqs. (A.12) and (A.13) do not apply since the response vector $\mathbf{Y}(t)$ is no longer a sub-Gaussian one. Unfortunately, the use of Eqs. (A.9) and (A.10) is not so straightforward as in the linear case, because closed-form solutions in terms of response PDF or CF for MDOF non-linear systems subject to the Gaussian input $a^{1/2}\mathbf{G}(t)$ are very rare. In this regard, it is worth mentioning that exact solutions are available for the stationary response PDF of some MDOF non-linear systems driven by external and/or parametric Gaussian white noise excitations[5,28]. In any case, if the underlying Gaussian vector process $\mathbf{G}(t)$ in Eq. (A.1) collects Gaussian white noises, i.e. $\mathbf{G}(t)\equiv\mathbf{W}_0(t)$, the powerful tools of the Itô stochastic differential calculus can be used to obtain the statistics of the response vector $\hat{\mathbf{Y}}(t)$ to the input $a^{1/2}\mathbf{W}_0(t)$ (see Eq. (A.2)), as required by the proposed approach. Specifically, the PDF, $p_{\hat{\mathbf{Y}}}(\mathbf{y};a,t)$, can be evaluated as approximate or exact solution (if it does exist) of the FPK equation:

$$\frac{\partial p_{\hat{\mathbf{Y}}}(\mathbf{y};a,t)}{\partial t}=-\boldsymbol{\nabla}_{\mathbf{y}}^{\mathrm{T}}(\mathbf{f}(\mathbf{y},t)p_{\hat{\mathbf{Y}}}(\mathbf{y};a,t))+\frac{a}{2}\boldsymbol{\nabla}_{\mathbf{y}}^{\mathrm{T}[2]}(p_{\hat{\mathbf{Y}}}(\mathbf{y};a,t))\mathbf{q} \tag{A.14}$$

where $\boldsymbol{\nabla}_{\mathbf{y}}^{\mathrm{T}}=\left[\partial/\partial y_1,\ \partial/\partial y_2,\ \ldots,\ \partial/\partial y_n\right]$; the exponent into square brackets means Kronecker power[30,31]; $\mathbf{q}$ is a n^2-vector whose elements are the strengths of the white noises $W_{0,j}(t)$. On the other hand, the CF, $\phi_{\hat{\mathbf{Y}}}(\boldsymbol{\vartheta};a,t)$, is ruled by the following partial differential equation:

$$\frac{\partial\phi_{\hat{\mathbf{Y}}}(\boldsymbol{\vartheta};a,t)}{\partial t}=\mathrm{i}\boldsymbol{\vartheta}^{\mathrm{T}}E[\exp(\mathrm{i}\boldsymbol{\vartheta}^{\mathrm{T}}\hat{\mathbf{Y}})\mathbf{f}(\hat{\mathbf{Y}},t)]-\frac{a}{2}\phi_{\hat{\mathbf{Y}}}(\boldsymbol{\vartheta};a,t)\boldsymbol{\vartheta}^{[2]\mathrm{T}}\mathbf{q}. \tag{A.15}$$

If the j-th element of the drift vector $\mathbf{f}(\hat{\mathbf{Y}},t)$ is a polynomial of the type:

$$f_j(\hat{\mathbf{Y}},t)=\sum_{k=1}^{m}\mathbf{c}_{jk}^{\mathrm{T}}\hat{\mathbf{Y}}^{[k]} \tag{A.16}$$

then, recalling that the following relationship:

$$E[\hat{\mathbf{Y}}^{[k]}\exp(\mathrm{i}\boldsymbol{\vartheta}^{\mathrm{T}}\hat{\mathbf{Y}})]=(-\mathrm{i})^k\boldsymbol{\nabla}_{\boldsymbol{\vartheta}}^{[k]}(\phi_{\hat{\mathbf{Y}}}(\boldsymbol{\vartheta};a,t)) \tag{A.17}$$

holds, Eq. (A.15) may be rewritten in terms of the unknown CF $\phi_{\hat{\mathbf{Y}}}(\boldsymbol{\vartheta};a,t)$ and its partial derivatives.