FRACTIONAL CALCULUS APPLICATION TO VISCO-ELASTIC SOLIDS

Di Paola M. [*]**, Pirrotta A.** [*]

[*] Dipartimento di Ingegneria Strutturale, Aerospaziale e Geotecnica (DISAG)
Università di Palermo
Viale delle Scienze, 90128 Palermo, Italy
e-mail: mario.dipaola@unipa.it

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Abstract. *It is widely known that fractional derivative is the best mathematical tool to describe visco-elastic constitutive law. In this paper it is shown that as soon as we assume the creep compliance function as power law type, as in the linearized version of the Nutting equation, then the fractional constitutive law appears in a natural way. Moreover, using Nutting equation for the creep function, the relaxation modulus is also of power law type whose coefficients (intensity and exponent) are strictly related to those of the creep compliance. It follows that by a simple creep test (or relaxation test) by means of a best fitting procedure we may easily evaluate the parameters of Nutting equation and then the fractional differential equation.*

1 INTRODUCTION

The theory of derivatives of non-integer order goes back in 1695 when L'Hospital asked Leibniz the significance of $\left(d^{1/2} f\left(x\right)/dx^{1/2}\right)$. After some attempts on properly defining fractional derivative due to Lacroix (1819), Fourier (1822), Abel (1823), Liouville (1832) and others, the most important step has been performed by Riemann (when he was a student) in 1847. It has to be stressed that fractional calculus is a misnomer since the order of the derivatives and integrals is not fractional but it belongs to the wider class of real number (or even complex ones), so that a more appropriate definition could be: generalized derivatives and integrals. However for historical reasons it is preferable not to change the nomenclature since all people refer to generalized differential calculus as fractional one.

Fractional operators are simply convolution integrals with power law kernel. The beauty of such operators is that they exactly behave as ordinary derivatives and integrals, that is all the rules of classical operators with integer order hold true, including Leibniz rule and integration by parts. Moreover also in Fourier and in Laplace domain the rules are quite similar and simple like the case of the classical derivatives and integrals of integer order. In the last century many applications have been performed to physics, chemistry, mechanics showing the versatility of such a calculus and simplifications to describe real-world problems. Then a question about so limited diffusion of such a powerful tool in the engineering community arises.

The answer to this question relies on many points: i) it is universally known that integer order derivatives have a clear geometrical and physical meaning. Fractional operators lack geometrical and physical interpretation. The question has been posed in many international conferences and round tables in which comes out that such a point is still an open problem. Even though attempts on this direction have been performed (see e.g. Podlubny[1] and references cited herein); ii)there are so many fractional operators like Marchaud, Riesz, Caputo, Grunewald, Letnikov, etc.. It follows that people may be disconcerted and it may be, to fell fear since the books on fractional calculus are voluminous and at first glance one may ask the motivation of so many different definitions. The reason of this relies on the fact that real problems require different operators. As an example in unbounded domain working in dynamics at steady state since the causality condition is irrelevant the Riesz operator is more amenable to get simpler solutions. The common point to all the definitions of fractional operators is the power law type of the kernel and that for every functional operator all the rules of classical calculus still hold for every kind of definitions. Our suggestion is let start only with Riemann-Liouville (RL) fractional operators and when a physical problem does not exactly overlap the RL definition give a look into the classical books and use the appropriate operator; iii) fractional derivatives and integrals may not be tackled by hands, avoid to conjecture fractional derivative. Such an example the fractional derivative of a constant is not zero. Our suggestion is first declare what a kind of fractional operator you use, second search in the books or simply use MATHEMATICA environment and you get the correct answer; iv) there are different symbologies to define the same operator, it follows that reading different paper often immediately you may not recognize the operator at hand.

Aim of this paper is to introduce the main definition of fractional calculus and the related properties. Our choice is to introduce only RL fractional operators in order to get a clue on this subject. In order to show the capabilities and the use of fractional calculus a very simple application on visco-elastic materials is presented since on this topic starting from the second part of the last century a lot of research effort has been addressed to asses the capability of the fractional calculus on this matter. It is widely known that the various single model of springs and dashpots to describe the visco-elastic model like Maxwell, Kelvin Voigt, Burger model involving first order derivatives do not fit experimental data. Then various attempts to use fractional derivatives has been made in the past. From the beginning of the last century it was shown that if the creep compliance, or relaxation are assumed to be a power law then with very few parameters an impressive coincidence with experimental data is reached.

Based on this observation in this paper it is shown that as soon as the creep compliance exhibits a power law decay, then the fractional derivative appears in a natural way and the various parameters (intensity and order of fractional operator) can be directly evaluated by a best fitting procedure on creep test.

2 FRACTIONAL CALCULUS

The simplest way to define fractional calculus is in considering this primitive of a function $f(x)$, we define such a primitive as $\left(I_{a^+}^1 f\right)(x)$, that is

$$\left(I_{a^+}^1 f\right)(x) = \int_a^x f(\xi) d\xi \qquad ; \qquad x > a \tag{1}$$

Generalization of eq.(1) leads to define the as $\left(I_{a^+}^n f\right)(x)$

$$\left(I_{a^+}^n f\right)(x) = \int_a^x \cdots \int_a^t f(\xi) d\xi \cdots dt \tag{2}$$

That is the n[th] primitive of a function is simply a n[th] integral. On the other hand in virtue of the Cauchy formula the n[th] primitive of $f(x)$ may be evaluated in the form

$$\left(I_{a^+}^n f\right)(x) = \frac{1}{(n-1)!} \int_a^x \frac{f(\xi)}{(x-\xi)^{1-n}} d\xi \qquad n \in \mathbb{N} \tag{3}$$

Eq.(3) shows that the multiple integral may be easily evaluated by a simple convolution integral, whose kernel is of power law type. The natural extension to a fractional integral is by considering an exponent $\alpha \in \mathbb{R}^+$. Since $\Gamma(\alpha)$ interpolate the factorial ($\Gamma(n) = (n-1)!$) then the proper extension of the Cauchy formula simply reads

$$\left(I_{a^+}^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(\xi)}{(x-\xi)^{1-\alpha}} d\xi \qquad ; x > a; \; \alpha \in \mathbb{R}^+ \tag{4}$$

That is known as Left Riemann-Liouville (RL) fractional integral. The right Riemann-Liouville (RL) fractional integral is then defined as

$$\left(I_{b^-}^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(\xi)}{(\xi-x)^{1-\alpha}} d\xi \qquad ; x < b; \; \alpha \in \mathbb{R}^+ \tag{5}$$

Once RL fractional integrals have been defined, the concept of RL fractional derivatives comes out in a natural way. In order to show this, we define the derivative of order m, denoted as $\left(\mathcal{D}^m f\right)(x) = d^m f(x)/dx^m$ in the form $\left(\mathcal{D}^{m+n}\left[\left(I^n f\right)(x)\right]\right) = \left(\mathcal{D}^m f\right)(x)$. It follows that the Riemann-Liouville fractional derivative may be defined as

$$\left(\mathcal{D}^{\alpha}_{a^+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \frac{d^m}{dx^m} \int_a^x \frac{f(\xi)}{(x-\xi)^{1-\alpha}} d\xi \tag{6a}$$

$$m = [\alpha] + 1$$

$$\left(\mathcal{D}^{\alpha}_{b^-} f\right)(x) = \frac{1}{\Gamma(\alpha)} \frac{d^m}{dx^m} \int_x^b \frac{f(\xi)}{(\xi - x)^{1-\alpha}} d\xi \tag{6b}$$

In eq.(6) the symbol $[\cdot]$ means integer part of the coefficient in the square brackets. RL fractional integrals and derivatives remain meaningful in unbounded domain putting $a \to -\infty, b \to \infty$. In the latter case we simply denote $\left(I^{\alpha}_{+} f\right)(x)$, $\left(\mathcal{D}^{\alpha}_{+} f\right)(x)$ for the left RL fractional integral and derivative respectively and $\left(I^{\alpha}_{-} f\right)(x)$, $\left(\mathcal{D}^{\alpha}_{-} f\right)(x)$ for the right RL fractional integral and derivative respectively. In literature the RL operators in unbounded domain are termed as Weyl operators. From the above definitions it follows that RL fractional integrals and derivatives are neither else than convolution integrals whose kernel is of power law type $(x-\xi)^{\alpha-1}$ interpolating derivatives and integral of integer order. At this point we may wonder the reason for calling such convolution operators fractional integrals and derivatives instead of convolutions with power law kernel. The reason is that it may be demonstrated that all the rules of classical differential calculus still hold for RL and Weyl fractional operators including Leibniz rule and integration by parts [2,3] this statement is of fundamental importance to operate with fractional calculus.
Such an example

$$\begin{array}{lll} \mathcal{D}^{\alpha}_{a^{\pm}}\left(\mathcal{D}^{\beta}_{a\pm} f\right)(x) = \left(\mathcal{D}^{\alpha+\beta}_{a\pm} f\right)(x) & ; & I^{\alpha}_{a^{\pm}}\left(I^{\beta}_{a\pm} f\right)(x) = \left(I^{\alpha+\beta}_{a\pm} f\right)(x) \\ \mathcal{D}^{\alpha}_{a^{\pm}}\left(I^{\beta}_{a\pm} f\right)(x) = \left(I^{\alpha-\beta}_{a\pm} f\right)(x) & ; & \alpha > \beta \\ \vdots & & \end{array} \tag{7}$$

Another fundamental remark that open the way to very interesting results is the fact that eqs.$4 \div 7$ remain meaningful for complex order $\gamma = \rho + i\eta$ ($\gamma = \rho \in \mathbb{R}^+, \eta \in \mathbb{R}, i = \sqrt{-1}$). In this case the integer value of the derivative in eqs.(6) has to be rewritten as $m = [\rho] + 1$.

3 FOURIER TRANSFORM OF RL FRACTIONAL INTEGRALS AND DERIVATIVES

Let us define the Fourier transform of a function $f(x)$ as

$$\Im\left[f(x);\vartheta\right] = \int_{-\infty}^{\infty} exp[i\vartheta x] f(x) dx = \hat{f}(\vartheta) \tag{8}$$

where $\hat{f}(\vartheta)$ is the Fourier transform of $f(x)$. Integration by parts lead to affirm that for Fourier transformable functions the following relationships

$$\Im\left[\left(\mathcal{D}^m f\right)(x);\vartheta\right] = (-i\vartheta)^m \hat{f}(\vartheta); \Im\left[\left(I^m f\right)(x);\vartheta\right] = (i\vartheta)^m \hat{f}(\vartheta) \quad (9)$$

hold true. Moreover it may be easily demonstrated that for Fourier transformable functions we have

$$\Im\left[\left(\mathcal{D}_{\pm}^{\gamma} f\right)(x);\vartheta\right] = (\mp i\vartheta)^{\gamma} \hat{f}(\vartheta) \ ; \ \Im\left[\left(I_{\pm}^{\gamma} f\right)(x);\vartheta\right] = (\pm i\vartheta)^{\gamma} \hat{f}(\vartheta) \ ; \quad \gamma = \rho + i\eta; \ \ \rho > 0 \quad (10)$$

This very interesting result shows that RL fractional operators behave like the Fourier transform of classical derivatives and integrals. Analogous results are obtained for Laplace transforms[3], that is, considering $F(s)$ the Laplace transform of $f(t)$ as

$$F(s) = L\left(f(t)\right) = \int_0^{\infty} e^{-st} f(t)\,dt \quad ; \quad s \in \mathbb{C} \quad (11)$$

Laplace transforms $L(\cdot)$ of fractional integrals and derivative are given in the form

$$L\left(I_{0^+}^{\alpha} f\right)(t) = s^{\alpha} F(s) \quad ; \quad L\left(D_{0^+}^{\alpha} f\right)(t) = s^{-\alpha} F(s) \quad (12)$$

4 PITFALLS IN THE CLASSICAL THEORY OF VISCO-ELASTICITY

Let us start with the derivation of the Duhamell integral of the simplest linear system that is the Kelvin visco-elastic model shown in Fig.(1)

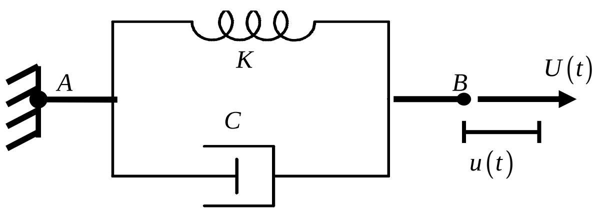

Figure 1. Kelvin model of visco-elastic system

Such a model is constituted by a spring with constant K and a dashpot characterized by a damping coefficient C and then the force exerted by the spring is $Ku(t)$ and the force exerted by the dashpot is $C\dot{u}(t)$, being $u(t)$ and $\dot{u}(t)$ the relative displacement and velocity between A and B, respectively. Now let us consider a De Saint Venant visco-elastic bar of cross A in traction with quasi-static load $N(t)$, each elementary volume is loaded by $\sigma(t)\,dx\,dy$ (Fig.(2)). By denoting as E the elastic modulus and c the viscosity coefficient we get for each elementary volume

$$\sigma(t) = E\frac{du(t)}{dz} + c\frac{d\dot{u}(t)}{dz} \tag{13}$$

Then by denoting with $du(t)/dz$, the strain $\varepsilon(t)$, the constitutive law, for the visco-elastic Kelvin model reported in Fig.1,is simply rewritten as

$$\sigma(t) = E\varepsilon(t) + c\,\dot{\varepsilon}(t) \tag{14}$$

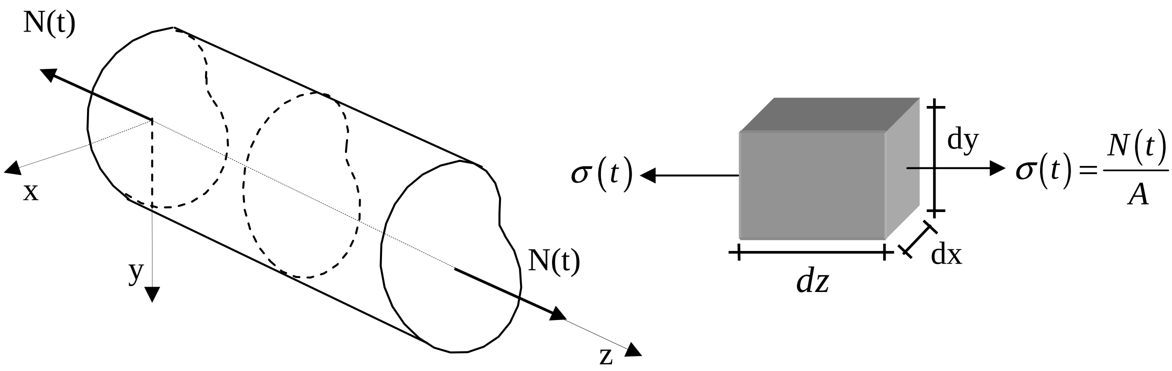

Figure 2. De Saint Venant visco-elastic bar in traction and an elementary volume

Inspection of eq.(14) reveals that constitutive law is ruled by ordinary differential equation in the unknown $\varepsilon(t)$. On the other hand the response to an unitary external load $N(t) = U(t)$, ($U(t) = 1, \forall t \geq 0; U(t) = 0, \forall t < 0$), denoted as $J(t)$ is given as

$$J(t) = \left[\frac{1}{E}\left(1 + \exp[-\beta t]\left(-1 + EJ(0)\right)\right)\right] \tag{15}$$

Being $\beta = E/c$ and $J(0)$ the value in $t = 0$ for an unitary stress load. $J(t)$ is termed in literature as creep compliance function. It is obvious that the impulse response function is the derivative of eq.(15), that is

$$h(t) = \dot{J}(t) = \frac{1}{c}\exp[-\beta t]\left(1 - EJ(0)\right) \tag{16}$$

It follows that the response to an arbitrary load history $\sigma(t)$ is given in the Duhamell form

$$\varepsilon(t) = \frac{1}{c}\int_0^t \dot{J}(t-\tau)\sigma(\tau)\,d\tau \tag{17}$$

Analogous result may be found using Boltzmann's superposition principle[4]. The convolution integral in eq.(17) is a hereditary one in the sense that the actual displacement (due to the external load) depends on the entire past load history $\sigma(t)$.

At this stage it is necessary to make some comments: i) the creep compliance function does not come in a natural way because of the presence of $J(0)$ that affects the result as shown in eq.(17). Following the physics of the problem $J(0)$ is assumed to be the instantaneous elastic response and then $J(0)=U(t)/E$; ii) the solution of eq.(15) performed step-by-step only requires the actual value of the load in t_j and the knowledge of the state variable $\varepsilon(t_j)$ in order to predict the state variable in $t_j+\Delta t$; iii) the kernel in the Duhamell integral, coming from a linear differential equation is of exponential type. The latter behaviour happens for more refined models like Burgers model[5] or any other various combinations of spring and dashpot like generalized Maxwell or Kelvin models; this is due to the fact that such refined models are governed by linear differential equations, and the homogeneous solution of such equations remains of the type $\sum_{k=1}^{m} a_k \exp(\lambda_k t)$ being λ_k the eigenvalues of such ordinary differential equations and m the number of dashpots present in the model ; iv) observation of Fig.(3) in which the creep compliance of the Kelvin model reveals that in $t=0$ there is a slope discontinuity in $t=0$ and for time $t\to\infty$ there is an asymptotic value. More refined models based on various combinations of springs and dashpots always show similar behaviour that do not follow experimental evidence.
None of these conclusions in using linear differential equations strictly follow experimental tests.

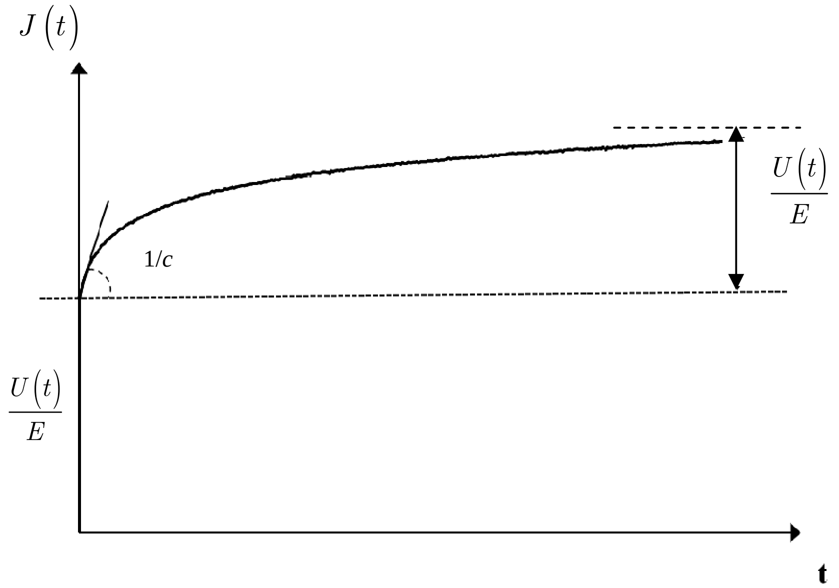

Figure 3. Creep compliance of the Kelvin model

On the other hand[6] Nutting (1921) from experimental data observed that for assigned $\sigma = const$, $(\forall t \geq 0)$

$$J(t) = b\,\sigma^r t^\alpha \tag{18}$$

that is known as Nutting equation for nonlinear visco-elasticity. In eq.(18) the parameters, b, r,α are determined from experimental tests. For $r=1$ we have the linear visco-elastic material. Scott-Blair[7], used eq.(18) for various soft materials (such as rubber) yielding a satisfactory description of the short-time creep data of these materials.

5 FRACTIONAL MODEL OF LINEAR VISCO-ELASTIC MATERIAL

In this section it will be shown that, by using the Nutting law to describe the stress-strain relation, the fractional model of linear visco-elasticity comes out in a natural way. It has to be stressed that such a model has been proposed by several authors. Scott Blair[7] was the first author that used fractional derivative in the constitutive law. The common motivation for describing the constitutive law using fractional derivatives is that, visco-elastic materials exhibit an intermediate behaviour between pure solid (stress is proportional to the strain $\varepsilon\left(t\right)$) and pure fluids (stress is proportional to $\dot{\varepsilon}\left(t\right)$) then such intermediate behaviour between pure solids and pure fluids may be described by an "intermediate" operator that is the fractional derivative as

$$\sigma\left(t\right) = E_\alpha\left(\mathcal{D}^\alpha_{0^+}\varepsilon\right)\left(t\right) \tag{19}$$

where E_α is an anomalous Young modulus ($\left[E_\alpha\right] = \left[Ft^{1-\alpha}/L^2\right]$).

This equation is known as Scott-Blair formula. Generalization of this formula to many parameters may be found in literature[8,9,10,11,12] , just to cite few. A pertinent bibliography on this topic may be found in the book of Podlubny[1] (1998). Once eq.(19) or a more refined models involving a summation of fractional derivatives at the right hand side of eq.(19) the problem relies on determining the parameters in the fractional differential equation. Hereinafter it will be shown that both from creep or relaxation tests the parameters may be found in a very simple way.

Let us start with the Nutting formula with $r = 1$ (linear visco-elastic model). The creep compliance function is neither else that the response of the system to a unit step function, then its derivative is the impulse response function (if we let $\sigma = 1,\ \forall t \geq 0$), that is

$$\frac{d\,J\left(t\right)}{dt} = b\alpha\,t^{\alpha-1} \tag{20}$$

then the Duhamell integral is written in the form

$$\varepsilon(t) = b\alpha\int_0^t \left(t-\tau\right)^{\alpha-1}\sigma\left(\tau\right)d\tau = \frac{b\alpha\Gamma(\alpha)}{\Gamma(\alpha)}\int_0^t \frac{\sigma(\tau)}{\left(t-\tau\right)^{1-\alpha}}d\tau \tag{21}$$

setting $\left(c_\alpha\right)^{-1} = \alpha b\Gamma\left(\alpha\right)$ eq.(21) may be rewritten in the form

$$\varepsilon(t) = \frac{1}{c_\alpha}\left(I^\alpha_{0^+}\sigma\right)\left(t\right) \tag{22}$$

the latter, using eq.(7) may be recast in the form

$$\sigma\left(t\right) = c_\alpha\left(\mathcal{D}^\alpha_{0^+}\varepsilon\right)\left(t\right) = E_\alpha\left(\mathcal{D}^\alpha_{0^+}\varepsilon\right)\left(t\right) \tag{23}$$

that coalesces with the Scott-Blair equation being $E_\alpha = c_\alpha$. Eq.(23) remains valid for a quiescent system at $t = 0$. In simple words, as soon as we assume the Nutting formula

specialized for the linear case (eq.18), then by using the classical definition of dynamics of Duhamell integral, the fractional derivative in the constitutive law comes out in a natural way. Now we are able to predict the parameters E_α, α from the test giving the creep compliance. That is from experimental test, by using a classical best fitting procedure b and α of the Nutting equation may be determined, in turn $E_\alpha = \left(\alpha b \Gamma(\alpha)\right)^{-1}$ may be determined as well. It is worth stressing that α remains the order of the fractional derivative.
On the other hand, often tests are performed giving the relaxation modulus $\Psi(t)$, that is the stress history due to an assigned (constant) strain. In this case the relaxation modulus for visco-elastic material is found to be well fitted by the power law equation

$$\Psi(t) = \Psi_0\, t^{-\beta} \qquad ;0<\beta<1 \quad ; \qquad [\Psi_0] = \frac{N s^\beta}{L^2} \tag{24}$$

Denoting with $J(s)$ and $\Psi(s)$ the Laplace transform of the creep compliance and relaxation modulus, respectively, it follows that the Laplace transform of eq.(24) is simply written as

$$\Psi(s) = \frac{\Psi_0}{s^{1-\beta}}\ \Gamma(1-\beta) \tag{25}$$

Since $J(s)$ and $\Psi(s)$ are related each another by the relationship[13]

$$\Psi(s) J(s) = \frac{1}{s^2} \tag{26}$$

then

$$J(s) = \frac{s^{-1-\beta}}{\Psi_0 \Gamma(1-\beta)} \tag{27}$$

whose Laplace transform gives

$$J(t) = \frac{t^\beta}{\Psi_0 \Gamma(1-\beta)\Gamma(1+\beta)} = b t^\alpha \tag{28}$$

being $b = \dfrac{1}{\Psi_0 \Gamma(1-\beta)\Gamma(1+\beta)}$ and $\alpha = \beta$. It follows that from the relaxation test, by means of a best fitting procedure, Ψ_0 and β are readily found and with the aid of the latter expression both b and α. For defining the creep compliance, remain determined. Notice that. for $\alpha = 0$ eq.(23) is written as $\sigma(t) = c_0 \varepsilon(t)$ that is the constitutive law of a spring (solid) is restored, while for $\alpha = 1$ eq.(23) is written as $\sigma(t) = c_1 \dot{\varepsilon}(t)$ that is the constitutive law of a dashpot (fluid) is restored. Eq.(23) obtained from experimental test with $\alpha \in \mathbb{R}$ is then an intermediate model between pure solid and pure fluid.

From results in this section we may state that as soon as from experimental test on the visco-elastic material in terms of creep or relaxation function is well fitted with a power law, then the hereditary integral for the representation of the response both in terms of stress or strains is a fractional operator.

6. CONCLUSIONS

It is widely understood that the constitutive law for visco-elastic solids involves fractional operators. The common motivation for introducing such operator is that visco-elastic materials exhibit an intermediate behavior between pure solid and pure fluid and then such an intermediate behavior is well fitted by using an intermediate operator that is the fractional operator. In this paper it is shown that by assuming the Nutting equation for the creep compliance function $J\left(t\right)$ ($J\left(t\right)=b\,\sigma^{r}t^{\alpha}$) then the constitutive law of the visco-elastic system is governed by a Riemann Liouville fractional operator, whose order α coalesces with the real exponent of the Nutting equation .

It is also shown that by using Nutting equation for the creep function, the relaxation modulus is also of power law type whose coefficients (intensity and exponent) are strictly related to those of the creep compliance. It follows that by a simple creep test (or relaxation test) by means of a best fitting procedure we may easily evaluate the parameters of Nutting equation and then the fractional differential equation.

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