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GALERKIN SCHEME BASED DETERMINATION OF FIRST-PASSAGE PROBABILITY OF BILINEAR SYSTEM WITH FRACTIONAL DERIVATIVE ELEMENT

P.D. Spanos[*], A. Di Matteo[†] and A. Pirrotta[†]

[*] Department of Mechanical Engineering and Materials Science
Rice University
6100 Main street, Houston, TX, USA
e-mail: spanos@rice.edu
[†] Dipartimento di Ingegneria Civile, Ambientale, Aerospaziale, dei Materiali (DICAM)
Università degli Studi di Palermo
Viale delle Scienze, 90128 Palermo, Italy
e-mail: alberto.dimatteo@unipa.it

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Abstract. *In this paper an approximate analytical technique is developed for determining the first-passage probability of randomly excited nonlinear hysteretic oscillator endowed with fractional derivatives elements. The amplitude of the system response is modeled as one-dimensional Markov process relying on a combination of stochastic averaging and statistical linearization technique. This yields the backward Kolmogorov equation which governs the evolution of the survival probability of the oscillator. An approximate solution of this equation is derived by resorting to a Galerkin approach. Specifically a convenient set of confluent hypergeometric functions, related to the corresponding linear oscillator with integer order derivatives, is used as orthogonal basis for this scheme. Application to the bilinear oscillator is presented. Comparison with pertinent Monte Carlo simulations demonstrates the accuracy of the proposed approximate analytical solution.*

1 INTRODUCTION

In dealing with the response determination of oscillators comprising fractional terms and subjected to stochastic excitations, a relevant problem, of high engineering interest, pertains to the so-called survival probability of the system; that is, the probability that its response will not reach a certain barrier over a given time interval. Several research efforts have then focused in the past decades on this topic, or its counterpart known as first-passage problem, that is the probability the system response reaches the prescribed barrier for the first time. Readers may refer to[1] for further details on recent approaches to address this problem.

In this regard, it is noted that the great majority of existing approaches for determining the first-passage probability consider systems with only integer order derivatives, while the first-passage problem of systems with fractional derivative elements is much less addressed. Specifically, in[2] the first-passage failure of fractional SDOF nonlinear systems under Gaussian white noise is solved through stochastic averaging method, while in[3] this technique

is adopted to deal with fractional derivative systems with power-form nonlinear restoring force.
In this paper a combination of stochastic averaging and statistical linearization is employed to yield a first order stochastic differential equation governing the response envelope of the original nonlinear oscillator with fractional derivative terms. In this way the backward Kolmogorov (BK) equation that governs the evolution of the survival probability is restored. An approximate solution of this equation is then derived through a Galerkin scheme approach, based on the orthogonality properties of the confluent hypergeometric functions[4-6]. Further, pertinent Monte Carlo simulation results are used to assess the accuracy of the proposed procedure. Results shows that the herein developed Galerkin based approach may represent a convenient alternative for determining the survival probability of linear and nonlinear systems with fractional derivatives elements, efficiently and with a satisfactory degree of accuracy.

2 MATHEMATICAL FORMUALATION

2.1 Markovian model of response amplitude

Consider a nonlinear single degree of freedom oscillator with fractional derivative elements, whose motion is governed by the following differential equation

$$\ddot{x}(t)+C_{\alpha}\left({}_{0}^{C}D_{t}^{\alpha}x\right)(t)+z(t,x,\dot{x})=w(t) \tag{1}$$

where a dot over a variable denotes differentiation with respect to time t; C_{α} is a constant which can be viewed as a damping coefficient if ($\alpha=1$), or as a stiffness coefficient ($\alpha=0$); $z(t,x,\dot{x})$ is an arbitrary nonlinear function of the response displacement and velocity; $w(t)$ is a zero mean Gaussian white noise process of power spectral density S_0 and $\left({}_{0}^{C}D_{t}^{\alpha}x\right)(t)$ is as α-order Caputo fractional derivative defined as[7]

$$\left({}_{0}^{C}D_{t}^{\alpha}x\right)(t)=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{\dot{x}(t-\tau)}{\tau^{\alpha}}d\tau,\quad 0<\alpha<1 \tag{2}$$

in which $\Gamma(\cdot)$ is the Gamma function.
Further Eq. (1) can equivalently be written as

$$\ddot{x}(t)+\beta_{0}\,\dot{x}(t)+z(t,x,\dot{x})+h\left[x(t),\dot{x}(t)\right]=w(t) \tag{3}$$

where $h\left[x(t),\dot{x}(t)\right]=C_{\alpha}\left({}_{0}^{C}D_{t}^{\alpha}x\right)(t)-\beta_{0}\dot{x}(t)$.
and $\beta_0=2\zeta_0\omega_0$ is an arbitrary chosen value of linear damping coefficient, being ω_0 the natural frequency of the corresponding linear oscillator and $\zeta_0\ll 1$ the ratio of critical damping.
In this way, the term containing the fractional derivative $h\left[x(t),\dot{x}(t)\right]$ can be assumed as a perturbation (not necessarily small) of an equivalent nonlinear perfectly viscoelastic system.
Assuming that $\zeta_0\ll 1$ and S_0 is $O(\zeta_0)$, it can be argued that the solution of Eq. (3) exhibits a pseudo-harmonic behavior, described by the following transformations of variables

$$x(t) = A(t)\cos\left[\omega(A)t + \theta(t)\right]$$
$$\dot{x}(t) = -A(t)\,\omega(A)\sin\left[\omega(A)t + \theta(t)\right] \tag{4}$$

where the amplitude *A(t)* and phase *θ(t)* are assumed to be slowly varying functions of time and the term $\omega(A)$ denotes an "effective" angular frequency of the system.

Squaring and summing both sides of Eqs. (4) yields

$$A(t) = \left[x^2(t) + \frac{\dot{x}^2(t)}{\omega^2(A)}\right]^{\frac{1}{2}} \tag{5}$$

Next, following a statistical linearization approach[1,7], a linearized counterpart of Eq. (3) can be given as

$$\ddot{x}(t) + \beta(A)\dot{x}(t) + \omega^2(A)x(t) + h\left[x(t), \dot{x}(t)\right] = w(t) \tag{6}$$

where the equivalent damping $\beta(A)$ and natural frequency $\omega(A)$ are assumed to be functions of the response amplitude $A(t)$, to account for the effect of the nonlinearity.

Performing a mean square minimization procedure on the error between Eqs. (3) and (6), the approximate expressions for $\beta(A)$ and $\omega(A)$ can be determined, assuming that the amplitude and phase remain constant over one period of oscillation. That is

$$\beta(A) = \beta_0 - \frac{1}{\pi A\omega(A)}\int_0^{2\pi} z\left[A\cos\phi - A\omega(A)\sin\phi\right]\sin\phi\, d\phi \tag{7}$$

and

$$\omega^2(A) = \frac{1}{\pi A}\int_0^{2\pi} z\left[A\cos\phi - A\omega(A)\sin\phi\right]\cos\phi\, d\phi \tag{8}$$

where $\phi(t) = \omega(A)t + \theta(t)$.

Further differentiating Eq. (5) with respect to time and taking into account Eqs. (4), yields

$$\dot{A}(t) = -\beta(A)A(t)\sin^2\phi(t) - \frac{h\left[A(t)\cos\phi(t), -A(t)\,\omega(A)\sin\phi(t)\right]}{\omega(A)}\sin\phi(t) - \frac{w(t)}{\omega(A)}\sin\phi(t) \tag{9}$$

A major task in solving Eq. (9) is related to the Caputo fractional derivative in terms of amplitude and phase which appears in the term $h\left[A(t)\cos\phi(t), -A(t)\,\omega_n(A)\sin\phi(t)\right]$.

However, following the derivations reported in[7], this problem can be circumvented. Specifically, relying on the assumption of light damping and utilizing a combination of deterministic and stochastic averaging, approximate decoupling the amplitude from the phase is achieved; the amplitude $A(t)$ is modeled as a one-dimensional Markov process governed by the differential equation

$$\dot{A}(t) = -\frac{1}{2}\beta(A)A(t) + \frac{\pi S_0}{2\omega^2(A)A(t)} + h(A)A(t) + \frac{\sqrt{\pi S_0}}{\omega(A)}\eta(t) \tag{10}$$

where $h(A) = \frac{\beta_0}{2} - \frac{C_\alpha}{2\omega^{1-\alpha}(A)}\sin\frac{\alpha\pi}{2}$, and $\eta(t)$ is a unitary intensity zero mean delta correlated stationary random process.

2.2 Backward Kolmogorov equation

Introducing the dimensionless variable

$$\bar{A}(t) = \frac{A(t)}{\sigma_s}; \quad \sigma_s = \sqrt{\frac{\pi S_0}{\beta_0 \omega_0^2}} \tag{11}$$

Eq. (10) can be rewritten in the form

$$\dot{\bar{A}}(t) = -\left\{\left[\frac{1}{2}\beta(\bar{A}) - \bar{h}(\bar{A})\right]\bar{A}(t) - \frac{\beta_0\omega_0^2}{2\bar{A}(t)\omega^2(\sigma_s\bar{A})}\right\} + \frac{\sqrt{\beta_0\omega_0^2}}{\omega^2(\sigma_s\bar{A})}\eta(t) \tag{12}$$

where σ_s represents the stationary standard deviation of the linear perfectly viscoelastic oscillator and

$$\bar{h}(\bar{A}) = \frac{\beta_0}{2} - \frac{C_\alpha}{2\,\omega^{1-\alpha}(\sigma_s\bar{A})}\sin\frac{\alpha\pi}{2} \tag{13}$$

$$\bar{\beta}(\bar{A}) = \beta_0 + \frac{S(\sigma_s\bar{A})}{\sigma_s\bar{A}\,\omega(\sigma_s\bar{A})} \tag{14}$$

Denoted by $P_B(a,t)$ the survival probability of $\bar{A}(t)$, that is, the probability that $\bar{A}(t)$ starting from an initial value *a* never reaches the barrier B during the interval [*0*,*t*]. Then, based on the Markovian approximation of $\bar{A}(t)$, it can be proved that $P_B(a,t)$ satisfies the following backward Kolmogorov (BK) equation

$$\frac{\partial P_B(a,t)}{\partial t} = -\left\{\left[\frac{1}{2}\bar{\beta}(a) - \bar{h}(a)\right]a - \frac{\beta_0\omega_0^2}{2a\,\omega^2(a)}\right\}\frac{\partial P_B(a,t)}{\partial a} + \left[\frac{\beta_0\omega_0^2}{2\omega^2(a)}\right]\frac{\partial^2 P_B(a,t)}{\partial a^2} \tag{15}$$

Taking into account the physics of the problem, the following initial and boundary conditions are imposed

$$\begin{aligned} &P_B(a,0) = 1; \quad 0 \le a \le B \\ &P_B(B,t) = 0 \\ &P_B(0,t) = \textit{finite} \end{aligned} \tag{16}$$

3 GALERKIN FORMULATION

As shown in[6], the BK equations corresponding to a linear oscillator with integer order derivatives ($\alpha \to 1$), along with the boundary conditions Eqs. (17), leads to a boundary value problem, which can be recast as a Sturm-Liouville one.

Let the variables $\lambda_{i,B}$ and $\Phi_{i,B}\left[E,\lambda_{i,B}\right]$ denote respectively the ith eigenvalue and the corresponding eigenfunction of the associated Sturm-Liouville problem of the corresponding linear oscillator with integer order derivatives ($\alpha \to 1$). As shown in[6], the ith eigenfunction can be found as

$$\Phi_{i,B}\left[E,\lambda_{i,B}\right] = M\left[-\lambda_{i,B},1,E\right] \tag{17}$$

where $E = \frac{1}{2}a^2$, and $M(\cdot)$ is the Krommer confluent hypergeometric function[4].

The corresponding eigenvalue $\lambda_{i,B}$ is the solution of the equation

$$M\left[-\lambda_{i,B},1,\frac{B^2}{2}\right] = 0 \tag{18}$$

Based on the properties of the eigenfunctions $\Phi_{i,B}\left[E,\lambda_{i,B}\right]$, an important orthogonality condition has also been derived as[6]

$$\int_0^{\frac{B^2}{2}} \Phi_{i,B}\left[E,\lambda_{i,B}\right]\Phi_{j,B}\left[E,\lambda_{j,B}\right]e^{-E}dE = 0, \quad i \neq j \tag{19}$$

Note that eigenvalues $\lambda_{i,B}$ in Eq. (18) have already been derived and tabulated[6] for several values of the barrier B.

The availability of the eigenvalues $\lambda_{i,B}$ and the orthogonality of the eigenfunctions $\Phi_{i,B}\left[E,\lambda_{i,B}\right]$ Eq. (17), suggested the use of this set of functions as a basis of a Galerkin scheme, for the determination of the survival probability of nonlinear systems without fractional derivative elements[1].

Here the same procedure is applied and extended to cope with nonlinear hysteretic systems endowed with fractional derivative elements.

Following the approach in[6,7] it can be argued that a solution of the BK equation Eq. (15), could be obtained by a Galerkin technique, using the already known eigenfunctions for the case, tabulated in[6] for several values of the barrier B. Specifically, an approximate solution of Eq. (15) is sought in the form

$$P_B(a,t) = \sum_{i=1}^{N} c_i(t)\Phi_{i,B}\left[E,\lambda_{i,B}\right] \tag{20}$$

where $c_i(t)$ are functions of time to be determined and N is an integer number which denotes the truncation limit of the series expansion.

Once eigenvalues $\lambda_{i,B}$ for the chosen value of the barrier B are obtained from the tables in[6] or from Eq. (18), eigenfunctions $\Phi_{i,B}\left[E,\lambda_{i,B}\right]$ can be directly determined through Eq. (17).

Thus, through this method, only functions $c_i(t)$ have to be determined in Eq. (20) for the solution of the problem Eq. (15). Specifically, omitting henceforth the dependence of the various variables and substituting Eq. (20) into Eq. (15) yields

$$\sum_{i=1}^{N} \dot{c}_i \Phi_{i,B} = -\left\{ \left[\frac{1}{2}\bar{\beta} - \bar{h} \right] a - \frac{\beta_0 \omega_0^2}{2a\,\omega^2(a)} \right\} \sum_{i=1}^{N} c_i \frac{d\Phi_{i,B}}{da} + \left(\frac{\beta_0 \omega_0^2}{2\omega^2} \right) \sum_{i=1}^{N} c_i \frac{d^2\Phi_{i,B}}{da^2} \tag{21}$$

Following the approach in[7] and taking into account the orthogonality condition Eq. (19), Eq. (21) yields the linear system

$$\dot{\mathbf{c}}(t) = \mathbf{\Psi}\mathbf{c}(t) \tag{22}$$

where $\mathbf{c}(t) = \left[c_1(t) \ldots c_N(t) \right]^T$, while $\mathbf{\Psi}$ is a $(N \times N)$ matrix whose elements are given by

$$\psi_{ij} = \frac{-\beta_0}{\int_0^{\frac{1}{2}B^2} \Phi_{i,B}^2 e^{-E} dE} \left\{ \lambda_j \int_0^{\frac{1}{2}B^2} \frac{\omega_0^2}{\omega^2} \Phi_{j,B} \Phi_{i,B}\, e^{-E} dE + \int_0^{\frac{1}{2}B^2} E \left(-\frac{\omega_0^2}{\omega^2} + \frac{\bar{\beta}}{\beta_0} - \frac{2\bar{h}}{\beta_0} \right) \frac{d\Phi_{j,B}}{dE} \Phi_{i,B} e^{-E} \right. \tag{23}$$

As far as the initial conditions of Eq. (22) are concerned, applying a similar approach as in[7], it can be proved that

$$c_i(0) = \frac{\int_0^{\frac{1}{2}B^2} \Phi_{i,B} e^{-E} dE}{\int_0^{\frac{1}{2}B^2} \Phi_{i,B}^2 e^{-E} dE}; \qquad i = 1, \ldots, N \tag{24}$$

Once Eq. (22), with the initial conditions in Eq. (24), is solved, then the survival probability $P_B(a,t)$ can simply be determined through Eq. (20), while the corresponding first-passage probability density function is obtained by using the equation

$$p_B(a,t) = -\frac{dP_B(a,t)}{dt} \tag{25}$$

4 NUMERICAL APPLICATION

In this section, the versatility and the accuracy of the proposed procedure is assessed considering the case of a single degree of freedom system with a fractional derivative element that exhibits hysteretic behavior of the bilinear type. The motion of the aforementioned system is governed by the differential equation

$$\ddot{x}(t) + C_\alpha \left({}_0^C D_t^\alpha x \right)(t) + \omega_0^2\, f\left[x(t), \dot{x}(t) \right] = w(t) \tag{26}$$

where $x(t)$ denotes the displacement relative response, ω_0 is the pre-yield natural frequency and the function $f\left[x(t),\dot{x}(t)\right]$, representing the oscillator restoring force, can be mathematically written as

$$f\left[x(t),\dot{x}(t)\right]=\gamma x(t)+(1-\gamma)z_0(t) \tag{27}$$

In this equation γ gamma is the post-yield to pre-yield stiffness ("rigidity") ratio, (e.g. the value $\gamma=0$ correspond to a perfectly elasto-plastic oscillator) and $z_0(t)$ is an auxiliary state variable governed by the differential equation

$$\dot{z}_0(t)=\dot{x}\left[1-H(\dot{x})H\left(z_0-x_y\right)-H(-\dot{x})H\left(-z_0-x_y\right)\right] \tag{28}$$

where x_y is the yielding displacement and $H(\bullet)$ is the Heaviside step function.

Taking into account Eqs. (9) and (10), the amplitude-dependent equivalent damping $\beta(A)$ and natural frequency $\omega(A)$ are respectively

$$\beta(A)=\beta_0+\frac{(1-\gamma)S(A)}{A\,\omega(A)};\quad \omega^2(A)=\gamma+\frac{(1-\gamma)C_0(A)}{A} \tag{29}$$

where

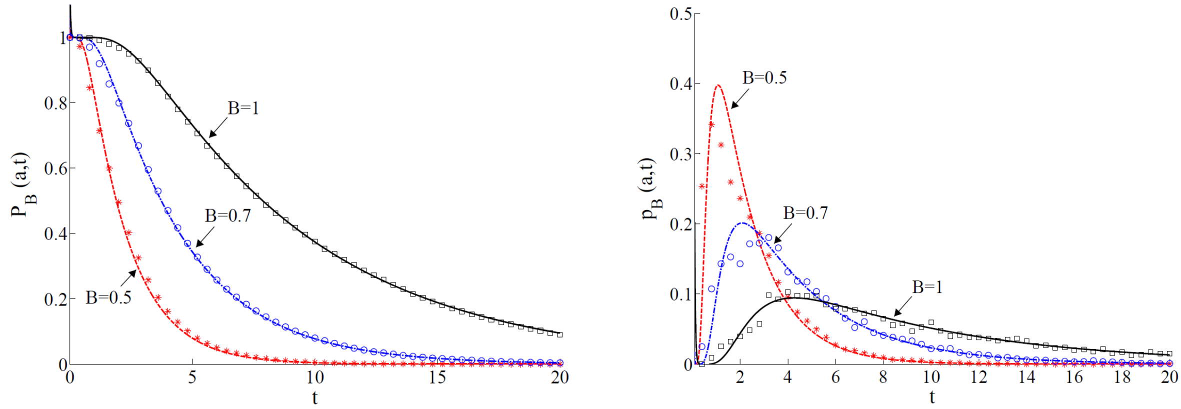
$$C_0(A)=\begin{cases}\dfrac{A}{\pi}\left[\theta^*-\dfrac{1}{2}\sin\left(2\theta^*\right)\right], & A>1\\ A, & A<1\end{cases};\quad S(A)=\begin{cases}\dfrac{4}{\pi}\left(1-\dfrac{1}{A}\right), & A>1\\ 0, & A<1\end{cases} \tag{30}$$

and $\cos\left(\theta^*\right)=1-\dfrac{2}{A}$.

Following the procedure described in the previous section, the coefficients of the matrix $\mathbf{\Psi}$ can be found, and Eq. (22) can be solved. Thus, the survival probability $P_B(a,t)$ can be determined through Eq. (20), while the corresponding first-passage probability density function $p_B(a,t)$ is obtained by using Eq. (25).

The Galerkin scheme formulation is applied to the oscillator in Eq. (26) possessing the parameter values $C_\alpha=0.05; \alpha=0.4; \omega_0=1; x_y=0.1$ and considering $\dfrac{\pi S_0}{\beta_0\,\omega_0^2}=1$. A value of the nonlinearity parameter $\gamma=0.8$ is used, while $N=11$ terms are used in the Galerkin scheme. Oscillators are considered initially at rest, that is $x(0)=0, \dot{x}(0)=0$ and hence $a=0$.

In Fig. 1 Galerkin scheme-based survival probabilities and first-passage time PDFs for these two values of the nonlinearity parameter are compared with pertinent MCS results. Specifically, results for three different values of the barrier *B* are presented (*B=0.5, 0.7, 1*). As shown in these figure, comparison to MCS data shows a quite good agreement.



(a) (b)

Figure 1. Galerkin scheme (lines) vis-à-vis MCS results (symbols) for the bilinear oscillator with $\gamma = 0.8$: a) Survival probability; b) First-passage probability density function

5 CONCLUDING REMARKS

The survival probability of a bilinear hysteretic system with fractional derivatives element under Gaussian white noise excitation has been studied. The response amplitude envelope has been modeled as a one-dimensional Markov process. This has led to the associated backward Kolmogorov (BK) equation ruling the evolution of the survival probability. An approximate solution of the BK equation has been pursued resorting to a Galerkin scheme. Specifically, the available eigenfunctions of a corresponding liner system with integer order derivatives have been used as basis of the Galerkin expansion. Proposed Galerkin based survival probability vis-à-vis pertinent Monte Carlo simulation data have demonstrated the accuracy and reliability of the suggested procedure.

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