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DICAM

ESTIMATION OF RESIDUALS FOR THE HOMOGENIZED SOLUTION OF RANDOM MEDIA

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Abstract. *The convergence of the elastic coefficients residuals compared to the homogeneous solution, at varying size of the representative volume sample, is studied in this work in the case of random two-dimensional media. In particular, it is considered the case of the masonry material which can be considered a heterogeneous solid with two phases (stones or bricks and mortar). Moreover, in relation to the peculiarities related to its realization, the masonry presents a quasi-periodic structure. A particular procedure of numerical generation of wall portions with quasi-periodic micro-structure has been developed by varying not only the scale ratio but also the mechanical ratio, between the characteristics of the stone and mortar, and the geometrical ratio, relative to the stones' dimensions. Compared to the homogeneous solution, the convergence of residuals has been studied in terms of probability density function and static moments up to the second order. The influence of the several analyzed ratios has been highlighted.*

Sommario. *La convergenza dei residui dei coefficienti elastici rispetto alla soluzione omogenea, al variare delle dimensioni del campione volume rappresentativo, è studiata in questo lavoro con riferimento al caso di continui bidimensionali aleatori. Viene in particolare considerato il caso della muratura che può essere considerata un solido eterogeneo costituito da due fasi (pietre o mattoni e malta). Peraltro, in relazione alle evidente peculiarità connesse alla sua realizzazione, la muratura presenta una microstruttura quasi - periodica. Una particolare procedura numerica di generazione di porzioni di muratura con micro-struttura quasi periodica è stata messa a punto variando non solo il rapporto di scala ma anche quello meccanico, fra le caratteristiche del mattone e della malta, e quello geometrico, relativo alle dimensioni delle pietre. Rispetto alla soluzione omogenea, la convergenza dei residui è stato studiata in termini di funzione densità di probabilità e momenti statici fino al secondo ordine evidenziando l'influenza dei diversi rapporti presi in esame.*

1 INTRODUCTION

Random media are materials with non-homogeneous mechanical characteristics that can be studied by the method of homogenization. With this method, the differential equations that rule the problem related to mechanical response of these materials to external actions can be expressed by means of two scales, one associated with the macroscopic characteristics and the other associated with microscopic characteristics. The homogenized solution is the part of the response which depends only on macroscopic terms.

In the literature several paper are devoted to the homogenization; usually in these papers the dependence on the microstructure is accomplished by periodic or random coefficients. In the case of random coefficients the characteristics of the equivalent homogeneous material can be found by means of the representative volume element (RVE).

The masonry can regarded as a heterogeneous material (composed by two phase: stone or bricks and mortar) with a quasi-periodic microstructure. The homogenization approach permits to individuate the RVE and obtain the mechanical characteristics of the equivalent homogeneous material[1, 2].

Nevertheless, a question of interest, which received a more limited attention in literature, is the estimation of the relation between the dimensions of the selected RVE and the errors in estimation of equivalent mechanical characteristics.

From a mathematical point of view, let us consider an elliptic differential operators with random coefficient depending on small parameter, this is the case of several problems in elasticity for heterogeneous media with random texture.

Under adequate condition[3,4,5], it is known that, as the parameter scale $\varepsilon \to 0$, the operator converge to an averaged operator of the same form with non random coefficients (this type of convergence is known as G-convergence). The accuracy of the approximation has been studied[6,7,8] but it is very difficult to adapt the proposed results in case of special material as masonry which is characterized by a quasi-periodic texture.

In previous papers[9, 10] the authors dealt preliminary with the case of the beam with Young's modulus randomly varying along the axis. In the present work, the authors extend that approach to the case of a region of the plane with randomly varying mechanical characteristics.

The convergence of the homogenized solution is studied in terms of the parameters which characterize the microscopic scale (ratio of elastic moduli, concentration ratio, correlation length), by means of numerical simulations. The aim of the work is to estimate the size of the representative volume element that assure a prefixed value of the error.

2 A CASE STUDY FOR RANDOM MEDIA: THE QUASI-PERIODIC MASONRY

The masonries of historical building very often have a non-periodical arrangement of the constituents phases, brick/stones and mortar. Nevertheless, the mutual arrangement is not completely random, since some amount of regularity can be observed. As an example, in Fig. 1 two samples of masonry walls, found on historical building in central Italy, are shown.

(a) (b)

Figure 1: Samples of non-periodic masonries.

As can be noted, the stones have different dimensions, but those of roughly the same height are laid out to form horizontal rows. As a consequence, different rows have different height; nevertheless, the height of the stones of the same row have small oscillation around the mean for that specific row. The length of the stones is also variable, and in what follows it is assumed that the length of the stones are uncorrelated with their height: as can be seen in the image, the length-to-height ratio can assumes very different values.

We define as quasi-periodic masonry the arrangement of bricks/stones and mortar in such a way that horizontal stones of roughly the same height can be identified in the same row of the wall. Moreover the mortar joints have roughly the same thickness and the vertical joints are not vertically aligned when two adjacent rows are considered.

3 GENERATION OF QUASI-PERIODIC MASONRY SAMPLES

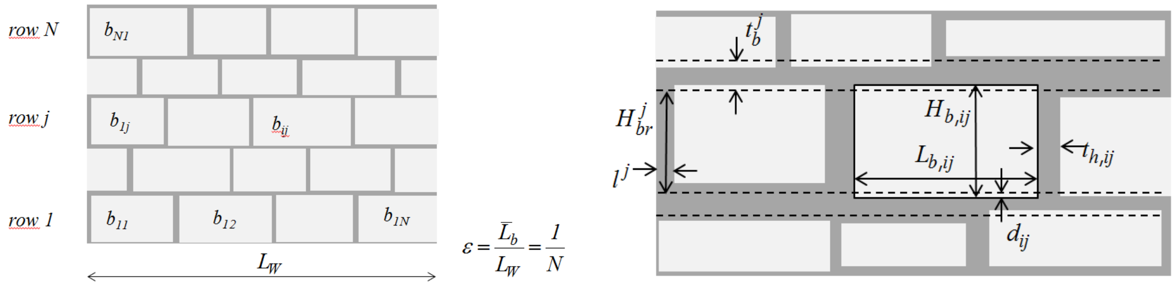
In order to perform a parametric analysis to assess the influence of both the mechanical and geometrical parameters of the constituent phases on the response of the masonry, several samples of non-periodic masonry wall have been generated. In particular, the samples have the fundamental characteristics of the non-periodic masonries (see Fig. 2a): (i) the masonry is made by rows of stones of almost the same height; (ii) the vertical joints of two adjacent rows are not aligned; (iii) the masonry joints have almost the same thickness.

The parameters which have been chosen for the different generations are:

- the length scale ratio $\epsilon = \bar{L}_b / L_W$, where $\bar{L}_b$ is the mean length of the stones and L_W is the length of the wall;

- the mechanical ratio E_b/E_m between the elastic moduli of stones and mortar;

- the geometrical ratio $\Delta L_b / \bar{L}_b = \Delta H_{br} / \bar{H}_b$, where ΔL_b is the variation of the length of the stones with respect to the mean length of the stones, $\bar{L}_b$, and ΔH_{br} is the variation of the mean of the height of the stones of a row with respect to the mean height of the stones $\bar{H}_b$.



(a) (b)

Figure 2: Generation of non-periodic masonry: (a) sample of non-periodic masonry; (b) parameters used for generation, example for stone *i* of row *j*.

The generated sample are made of N rows with N stone or $N+1$ stones (in this latter case the first and the last are really half stones). Therefore the length scale ratio can be defined also as $\epsilon = 1/N$ (see Fig. 2a). The probability density function (pdf) of the variation of the ΔL_b and ΔH_{br} has been assumed to be uniform. Anyway, we recall that the height and the width of the stones are generated with the hypothesis that they are uncorrelated. Moreover, also the thickness of head and bed mortar joints, t_h and t_b, is assumed to vary randomly according to an uniform pdf around the mean values $\bar{t}_h$ and $\bar{t}_b$. The values $\bar{L}_b$=300 mm, $\bar{H}_b$=150 mm, $\bar{t}_b = \bar{t}_h$=100 mm have been assumed. The geometry of the wall is generated by the following laws (refer to Fig. 2b for the quantities)

$$
\begin{aligned}
&L_{b,ij} = \bar{L}_b + U\left(-\tfrac{1}{2},\tfrac{1}{2}\right)\cdot \Delta L_b ,\\
&l^j = (-1)^j\left[\frac{1}{2}\left(\bar{L}_b + U\left(-\tfrac{1}{2},\tfrac{1}{2}\right)\cdot \Delta L_b\right) + \left(\bar{t}_h + U\left(-\tfrac{1}{2},\tfrac{1}{2}\right)\cdot \frac{\bar{t}_h}{3}\right)\right],\\
&H_{br}^j = \bar{H}_b + U\left(-\tfrac{1}{2},\tfrac{1}{2}\right)\cdot \Delta H_{br} , H_{b,ij} = H_{br}^j + U\left(-\tfrac{1}{2},\tfrac{1}{2}\right)\cdot \frac{\bar{t}_b}{3}, d_{ij} = U\left(-\tfrac{1}{2},\tfrac{1}{2}\right)\cdot \frac{\bar{t}_b}{3}\\
&t_b^j = \bar{t}_b + U(-0.5,0.5)\cdot \frac{\bar{t}_b}{3}, \ t_{h,ij} = \bar{t}_h + U(-0.5,0.5)\cdot \frac{\bar{t}_h}{3}
\end{aligned}
\tag{1}
$$

The assumed values of the length scale ratio, the mechanical ratio and the geometrical ratio used for the generations of the different sets are reported in Table 1. A total of 200 samples for each length scale ratio have been generated. Some examples of the obtained masonries are shown in Fig. 3.

set	1a	2a	3a	4a	5a	6a	1b	2b	3b	4b	5b	6b	1c	2c	3c	4c	5c	6c
ε	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$\frac{E_b}{E_m}$	10	5	3	10	5	3	10	5	3	10	5	3	10	5	3	10	5	3
$\Delta L_b/\bar{L}_b$ $\Delta H_{br}/\bar{H}_b$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Table 1: Length scale ratio, mechanical ratio and geometrical ration used for the generations.

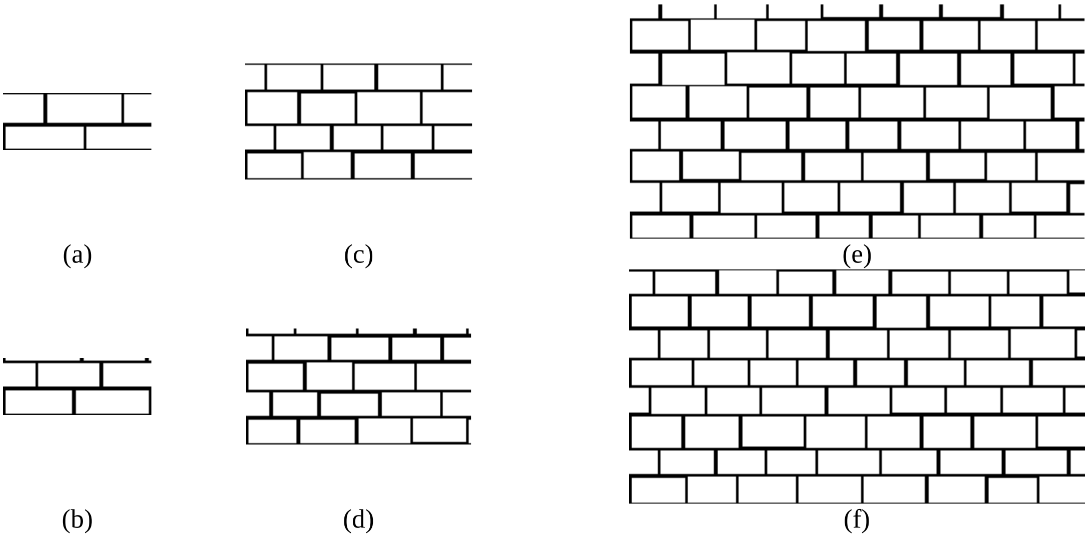


Figure 3: Samples of generated masonry walls: (a) e (b) samples with ε=0.5 (N=2); (c) e (d) samples with ε=0.25 (N=4); (e) e (f) samples with ε=0.125 (N=8).

For each sample, and for each set of properties reported in Table 1, the stiffness tensor has been evaluated. We recall that the stiffness tensor relates the mean over the domain V of stresses and strains

$$\begin{Bmatrix} \langle \sigma_{11} \rangle \\ \langle \sigma_{22} \rangle \\ \langle \sigma_{12} \rangle \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \cdot \begin{Bmatrix} \langle \varepsilon_{11} \rangle \\ \langle \varepsilon_{22} \rangle \\ \langle \varepsilon_{12} \rangle \end{Bmatrix}, \text{ with } \langle f \rangle = \frac{1}{V} \int_V f dV \qquad (2)$$

The value of the stresses and the strains is evaluated by means of a finite element analysis, where a four node plane stress element have been used.

In order to make comparisons, the stiffness tensor is also evaluated using a sample with much larger dimension (equivalent to $N=12$), which can be considered as the representative volume element: this tensor is referred to as the “homogenized” one.

4 RESULTS OF NUMERICAL SIMULATIONS

In the following some results obtained using the samples generated as described in the preceding section are presented. In particular, the results concern the obtained estimates for the component C_{11} of the stiffness tensor, but similar results hold also for the remaining components. In commenting the figures, it is worth recalling that the sets “set 4x” are those with greater mechanical and geometrical ratio while those “set 3x” have smaller ratio. Moreover “a” denotes the smallest samples and “c” the greatest samples in terms of dimension (or length scale ratio).

The results for the same mechanical and geometrical ratio and different length scale ratio are presented in Fig. 4. In this figure, c_1 is the concentration ration, which is defined as the percentage of the volume sample occupied by the stones. Moreover, the dashed line denotes the value for the homogenized masonry as defined previously, whose concentration ratio is 0.845. As can be observed, the estimates are less scattered as the dimension of the sample increases and as the mechanical characteristics ratio is smaller.

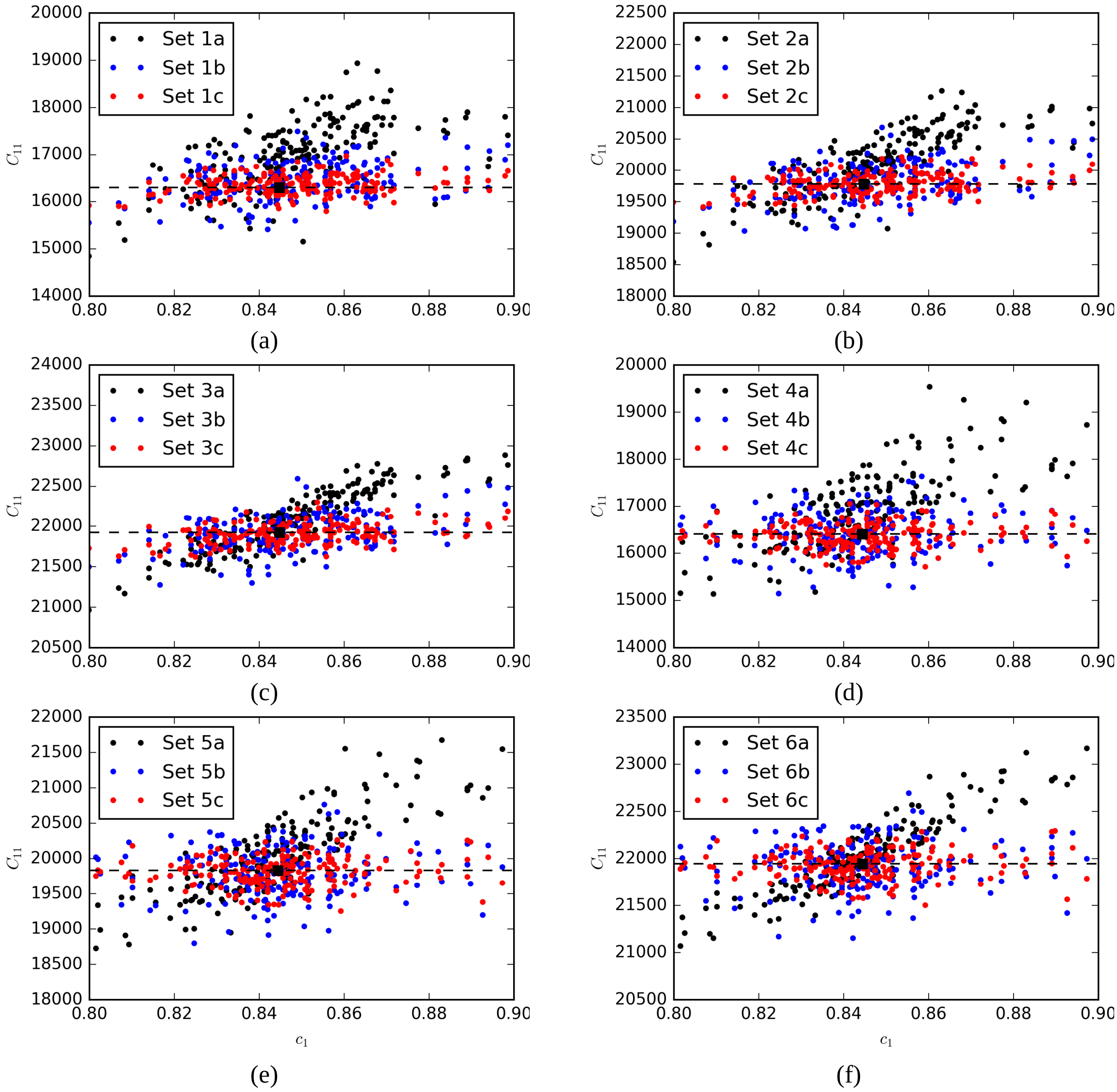


Figure 4: Obtained estimates for the 200 samples and for each combination of parameters.

The estimation of the probability density function is shown in Fig. 5. Again, it can be observed that the estimates are less scattered as the dimension of the sample increases and as the mechanical characteristics ratio are smaller.

Finally, the behaviour of the mean value of the estimated characteristic C_{11} and its coefficient of variation value are shown in Fig. 6. As can be observed, as the length scale ratio decreases (and therefore the dimension of the sample increases) the mean value tends to the homogenized value, while the coefficient of variation decreases.

These results allow to appreciate the rate of convergence and can be used to estimate the dimension of the RVE which permits to have an error under a prescribed value, in this contest the procedure proposed in a previous paper[10] can be used.

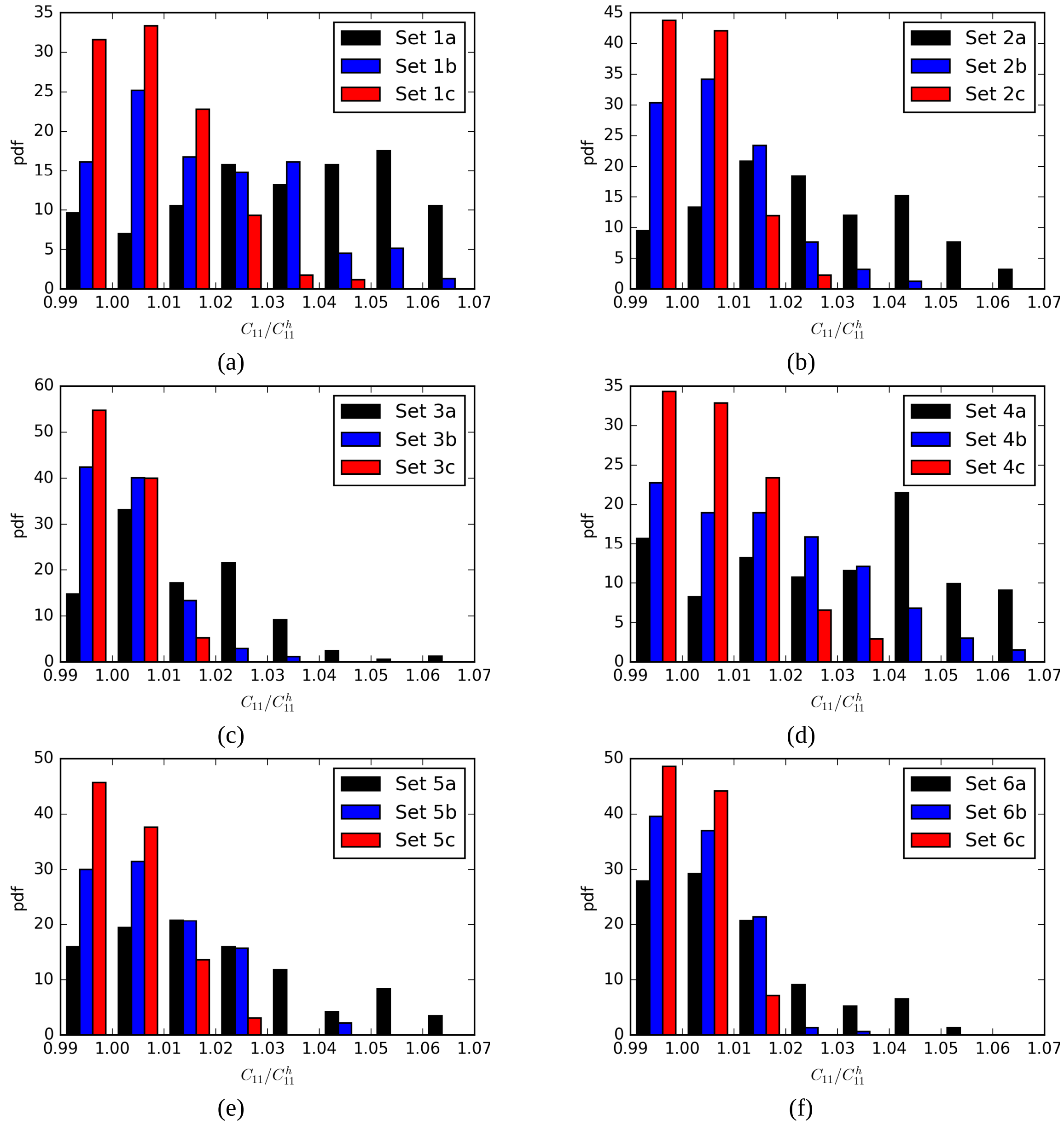


Figure 5: Estimation of probability density function for each combination of parameters.

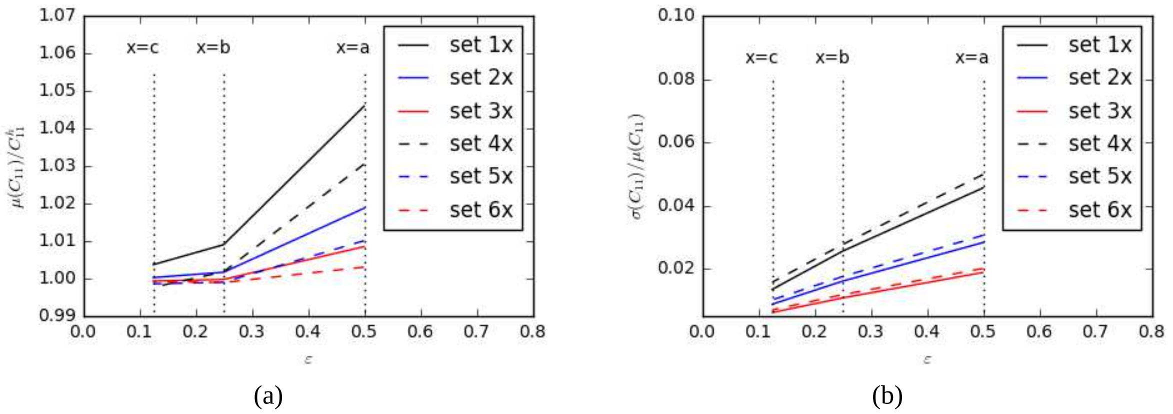


Figure 6: Effect of length scale ratio on the estimated mean and coefficient of variation of component C_{11}.

4 CONCLUSIONS

The present paper dealt with the analysis of the elastic coefficients residuals compared to the homogeneous solution in the case of heterogeneous two-dimensional media with random structure. Special attention is devoted to masonry material which can be considered as a solid composed by two phases (stones or bricks and mortar) with quasi-periodic micro-structure.

A specific numerical procedure has been developed to generate numerically masonry wall samples by varying the scale ratio. The convergence of the elastic moduli to the ones of the equivalent homogeneous continuum has been analyzed. The effects of the geometrical ratio, among the stones sizes, and mechanical ratio, between Young modulus of phases, have been taken into account.

The convergence to the homogeneous solution has been described in terms of probability density function and moments up to the second order. Even if further studies are necessary, the obtained results will permit to develop evaluation procedures to minimize errors in the detection of the representative volume element.

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