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PROBABILISTIC ANALYSIS OF THE RESPONSE OF FRACTIONAL VISCOELASTIC RAILTRACK UNDER RANDOM TRAIN SPEED

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Abstract. *In this paper the response analysis of the rail resting on the ballast modelled as a fractional viscoelastic material is presented. The train speed is modelled as a normal random variable with assigned mean and covariance. The probabilistic characterization of the response is performed by the state variable analysis method.*

1 INTRODUCTION

Nowadays, one of the most important issues in the railway field is to improve the strength, augment the track durability and increase the efficiency, especially for the heavier axle loads and high speed. In order to achieve such purpose it is necessary to have a better understanding of the static and the dynamic behaviour of the rail track structure by means of a proper mathematical model.

Since the beginning of the nineteenth century, the problem of moving loads is review in detail by Timoshenko in [1], due to the construction of the first railway bridges. In literature there exists several models for the rail track based on the theories of moving loads, by using different methods (analytical, semi-analytical, finite elements,...). These models are based on the theories of structures under moving loads presented by Fryba [2].

In order to improve the performance of the rail track, new materials and new configurations are now currently used, and exhibit pronounced viscoelastic behaviour. Classically viscoelastic materials have been model with different combinations of springs and dashpots. However, since Nutting experimental tests [3], it has been demonstrated the inconsistency of the classical models to characterize viscoelastic behaviour. Some years later, Scott Blair and Caffyn [4] proposed the use of fractional calculus (see [5]) to interpret mathematically Nutting experiments.

The aim of this paper is to model the rail track response under random train speed considering the rail track modelled as a simple supported elastic beam resting on a liner viscoelastic foundation (modelled with fractional constitutive laws) that simulates the ballast layer and the train is simulated as a moving force. The stochastic analysis is performed by using the state variable analysis method [6].

2 MODEL OF VISCOELASTIC BALLASTED RAIL TRACK

The rail track model consists in the rail and the ballast. The ballast layer may be made of asphalt, this material obeys the Nutting law [3] and then the constitutive (hereditary) law is well suited by a fractional operator. As in fact as the relaxation function *R(t)* of the ballast is given in the form

$$R(t)=\frac{\mathcal{E}_\beta}{\Gamma(1-\beta)}t^{-\beta} \tag{1}$$

where $\mathcal{E}_\beta$ and β are parameters depending o the material and $\Gamma(\cdot)$ is the gamma function. The Boltzman superposition principle gives the time history in terms of stress $\sigma(t)$ for an assigned strain history $\varepsilon(t)$ in the form

$$\sigma(t)=\int_0^t R(t-\tau)\dot{\varepsilon}(\tau)d\tau \tag{2}$$

Then by inserting Eq. (1) in Eq. (2) it is obtained

$$\sigma(t)=\frac{\mathcal{E}_\beta}{\Gamma(1-\beta)}\int_0^t (t-\tau)^{-\beta}\dot{\varepsilon}(\tau)d\tau=\mathcal{E}_\beta\left({}^C D_{0^+}^{\beta}\varepsilon\right)(t) \tag{3}$$

where the symbol $\left({}^C D_{0^+}^{\beta}\varepsilon\right)(t)$ stands for the Caputo fractional derivative. By performing experimental relaxation test on the material, the parameters $\mathcal{E}_\beta$ and β may be easily obtained by a best fitting procedure by assuming that Eq. (1) is valid. Then, by modelling the ballast as a viscoelastic Winkler model resting on the bedrock the force displacement relationship is written in the form

$$f(t)=\frac{E_\beta}{\Gamma(1-\beta)}\int_0^t (t-\tau)^{-\beta}\dot{u}(\tau)d\tau=E_\beta\left({}^C D_{0^+}^{\beta}u\right)(t) \tag{4}$$

where E_β is the new anomalous coefficient which account for the depth of the ballast layer and on the interdistance of the rails and on the interdistance of the sleepers, $u(\tau)$ is the vertical displacement of the top of the ballast and $f(t)$ is the load transmitted by the rail to the ballast. The governing equation of the rail is then given as

$$\rho A\ddot{u}(x,t)+EIu''''(x,t)+E_\beta\left({}^C D_{0^+}^{\beta}u\right)(x,t)=F(x,t) \tag{5}$$

where ρ, m, A, E, I, L are respectively the density, the mass, the area, the Young modulus, the inertial moment and the length of the rail. $F(x,\ t)$ in Eq. (5) is the force that the train applies in the rail. In the following it is supposed that the rail is so long that the train may be consider as a single load. In this case, $F(x,\ t)$ is given as

$$F(x,\ t)\ =M\ g\ \delta(x-xv(t))\ =M\ g\ \delta(x-vt), \tag{6}$$

where *M* is the total mass of the train, *g* is the gravity acceleration, *v* is the speed of the train and $\delta(\cdot)$ is the Dirac’s delta. For simplicity sake’s it is also supposed that the rails are simply supported. Since the system is linear the case in which it is necessary to take into account for single loads transmitted by the wheels then by using the superposition principle the analysis may be easily extended. In the Figure 1, a layout of the present model is represented.

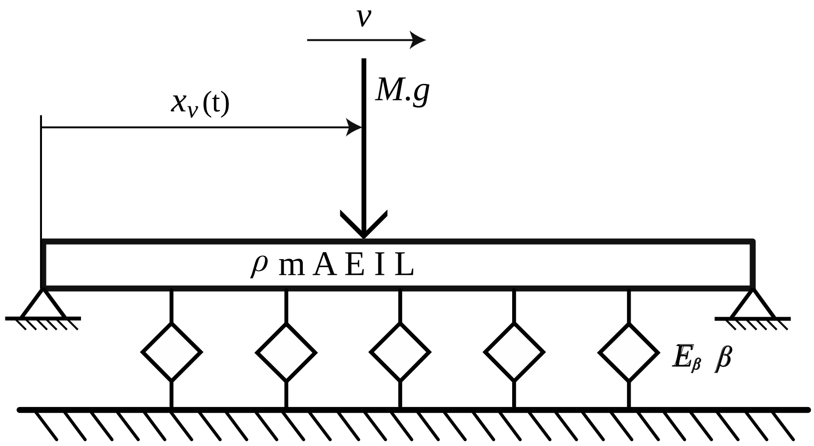

Figure 1: Ballasted rail track schema.

The vertical displacement $u(x,t)$ is calculated trough the Galerkin method, namely $u(x,t)$ is decomposed in the orthogonal basis $\psi_j(x)$ in the form

$$u(x,t)=\sum_{j=1}^{\infty}\psi_j(x)y_j(t) \tag{7}$$

where $\psi_j(x)=\sin(j\pi x/L)$. By inserting Eq. (7) into Eq. (5), multiplying by $\psi_k(x)$ and integrating in the domain (0, L), due to the orthogonality condition of the basis $\psi_j(x)$, the following set of fractional differential equations are obtained in the form

$$\ddot{y}_j(t)+EI\left(\frac{\pi j}{\rho AL}\right)^4 y_j(t)+\frac{E_\beta}{\rho A}\left({}^{C}D_{0^+}^{\beta}y_j\right)(t)=\frac{2Mg}{L\rho A}sin(j\frac{\pi vt}{L});\; j=1,2,\ldots\,\infty. \tag{8}$$

In the next section, the analysis method of such set of equations will be described.

3 METHODOLOGY: STATE VARIABLE ANALYSIS

Many solution method are available for solving the fractional differential equations, see [5,7], in this paper it is proposed the use of the state variable analysis method [6] that will be briefly summarised. Equation (8) may be rewritten in sequential form as follows

$$\sum_{k=1}^{n}C_{kj}({}^{C}D_{0^+}^{k\beta}y_j)+K_jy_j(t)=f_j(t), \tag{9}$$

where $n\beta$ is equal to the maximum order of derivation that appears in the given differential equation (in the case under study is two). And $C_{nj}=1$, $C_{1j}=(E_\beta/\rho A)$, $C_{kj}=0\,\forall k=2,3,\ldots,(n-1)$, $K_j=EI\left(\pi j/\rho AL\right)^4$ and $f_j(t)=(2Mg/L\rho A)sin(j\pi vt/L)$.

It is now introduced the state variable vector $\boldsymbol{z_j}$ in the form

$$\boldsymbol{z_j}^T(t)=\begin{bmatrix} y_j(t) & ({}^{C}D_{0^+}^{\beta}y_j)(t) & ({}^{C}D_{0^+}^{2\beta}y_j)(t) & \ldots & ({}^{C}D_{0^+}^{(n-1)\beta}y_j)(t)\end{bmatrix} \tag{10}$$

and appending the $(n-1)$ identities

$$\sum_{s=1}^{n-k}C_{s+k}\left({}^{C}D_{0^+}^{\beta}\,{}^{C}D_{0^+}^{(s-1)\beta}y_j\right)(t)=\sum_{s=1}^{n-k}C_{s+k}\left({}^{C}D_{0^+}^{s\beta}y_j\right)(t);\; k=1,2,\ldots,(n-1). \tag{11}$$

Eqs. (7) and (11) represent a set of n coupled differential equations readily cast in the form

$$\mathbf{A}_j\left({}^C D_{0^+}^{\beta}\boldsymbol{z}_j\right)(t)+\mathbf{B}_j\boldsymbol{z}_j(t)=\boldsymbol{g}_j(t), \tag{12}$$

where

$$\boldsymbol{g}_j^{\,T}(t)=[\begin{matrix} f_j(t) & 0 & 0 & \dots & 0\end{matrix}] \tag{13}$$

and $\mathbf{A}_j$ and $\mathbf{B}_j$ are symmetric matrices defined as

$$\mathbf{A}_j=\begin{bmatrix} C_{1j} & C_{2j} & \cdots & C_{(n-1)j} & C_{nj} \\ C_{2j} & C_{3j} & \cdots & C_{nj} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{(n-1)j} & C_{nj} & \cdots & 0 & 0 \\ C_{nj} & 0 & \cdots & 0 & 0 \end{bmatrix};\ \mathbf{B}_j=\begin{bmatrix} K_j & 0 & \cdots & 0 & 0 \\ 0 & -C_{2j} & \cdots & -C_{(n-1)j} & -C_{nj} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -C_{(n-1)j} & \cdots & 0 & 0 \\ 0 & -C_{nj} & \cdots & 0 & 0 \end{bmatrix} \tag{14}$$

$\boldsymbol{z}_j(t)$ can be decomposed in the orthogonal basis of the eigenvectors of $\mathbf{A}_j$ and $\mathbf{B}_j$ as follows

$$\Phi_j^T\mathbf{A}\Phi_j=\mathbf{U}_j;\ \Phi_j^T\mathbf{B}\Phi_j=\mathbf{V}_j \tag{15}$$

where $\mathbf{U}_j$ and $\mathbf{V}_j$ are diagonal matrices and Φ_j is the *(nxn)* matrix whose columns are the eigenvectors of the matrix $\mathbf{A}_j^{-1}\mathbf{B}_j$.

$$\mathbf{A}_j^{-1}\mathbf{B}_j=\frac{1}{C_{nj}}\begin{bmatrix} 0 & -C_{nj} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -C_{nj} \\ K_j & C_{1j} & \cdots & C_{(n-2)j} & C_{(n-1)j} \end{bmatrix} \tag{16}$$

By means of modal transformation

$$\boldsymbol{z}_j(t)=\Phi_j\boldsymbol{p}_j(t) \tag{17}$$

a new set of uncoupled fractional differential equation is derived in the form

$$\mathbf{U}_j({}^C D_{0^+}^{\beta}\boldsymbol{p}_j)(t)+\mathbf{V}_j\boldsymbol{p}_j(t)=\Phi_j^{\,T}\boldsymbol{g}_j(t), \tag{18}$$

by indicating with u_{jk} and v_{jk} the k-th diagonal element of the matrix $\mathbf{U}_j$ and $\mathbf{V}_j$ respectively, it follows

$$\left({}^C D_{0^+}^{\beta}p_{jk}\right)(t)+\xi_{jk}p_{jk}(t)=\tilde{\phi}_{j\ 1k}g_{jk}(t)=\tilde{\phi}_{j\ 1k}\ f_j(t) \tag{19}$$

where $\xi_{jk}=\dfrac{v_{jk}}{u_{jk}}$, $\tilde{\phi}_{j\ 1k}=\dfrac{\phi_{j\ 1k}}{u_{jk}}$ being $\phi_{j\ 1k}$ the component *(1,k)* of the matrix Φ_j. Now, it is necessary to calculate the first component of the vector $\boldsymbol{z}_j(t)$ and hence it is now written

$$y_j(t)\ =z_{j1}(t)\ =\sum_{k=1}^{n}\phi_{j\ 1k}p_{jk}(t) \tag{20}$$

In order to solve Eq. (19), it is discretized the time axis into small intervals of equal length Δt, and it is grouped $p_{jk}(\Delta t)$, $p_{jk}(2\Delta t)$, … , $p_{jk}(n\Delta t)$ into a vector labelled as

$$\mathbf{p_{jk}}^T=\left[\begin{matrix} p_{jk}(\Delta t) & p_{jk}(2\Delta t) & \rightleftharpoons & p_{jk}(n\Delta t)\end{matrix}\right] \tag{21}$$

and applying the Grünwald-Letnikov integration method it is obtained that the solution of Eq. (19) is given in the form

$$\left(\mathbf{D}+\xi_{jk}\mathbf{I}\right)\mathbf{p_{jk}}=\tilde{\phi}_{j\ 1k}\mathbf{q_j}(v) \tag{22}$$

where **I** is the identity matrix and the matrix **D** is the Grünwald-Letnikov operator of the fractional differential equation, that in matrix form is written as

$$\mathbf{D}=\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \lambda_2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{m-1} & \lambda_{m-2} & \cdots & 1 & 0 \\ \lambda_m & \lambda_{m-1} & \cdots & \lambda_2 & 1 \end{bmatrix} \tag{23}$$

where $\lambda_i=\lambda_{i-1}(j-1-\beta)/j$, $\lambda_1=1$ and $\mathbf{q_j}(v)$ is written as

$$\mathbf{q_j}^T(v)=\frac{2Mg}{L\rho A}\left[\ sin\left(j\Delta t\frac{\pi v}{L}\right)\quad sin\left(2j\Delta t\frac{\pi v}{L}\right)\quad \rightleftharpoons\quad sin\left(mj\Delta t\frac{\pi v}{L}\right)\ \right]. \tag{24}$$

4 STOCHASTIC ANALYSIS: RANDOM TRAIN SPEED

As it has been found in literature [8], it is assumed that the train speed is modelled as a normal random variable whose PDF is characterized by the mean value μ_v and the covariance σ_v, namely

$$p_v(v)=\frac{1}{\sqrt{2\pi}\sigma_v}e^{-\frac{(v-\mu_v)^2}{2\sigma_v^2}} \tag{26}$$

Sine the system is linear and the input is Gaussian, the response is also Gaussian and then its probabilistic characterization is fully described by the mean and the covariance.

The mean of $\mathbf{p_{jk}}$ is denoted as $E[\mathbf{p_{jk}}]$ and is calculated as follows

$$E[\mathbf{p_{jk}}]=\tilde{\phi}_{j\ 1k}\left(\mathbf{D}+\xi_{jk}\mathbf{I}\right)^{-1}E[\mathbf{q_j}]=\mathbf{Q}_{jk}E[\mathbf{q_j}(v)] \tag{27}$$

where $\mathbf{Q_{jk}}=\tilde{\phi}_{j\ 1k}\left(\mathbf{D}+\xi_{jk}\mathbf{I}\right)^{-1}$.

In order to calculate $E[\mathbf{p_{jk}}]$ it is necessary to calculate the mean of the vector $\mathbf{q_j}(v)$, defined as

$$\mu_{\mathbf{q_j}}=E[\mathbf{q_j}(v)]=\int_{-\infty}^{\infty}\mathbf{q_j}(v)p_v(v)dv \tag{28}$$

Then by inserting Eq. (26) in Eq. (28) the r-th component of the vector $\mathbf{q_j}(v)$ labelled as q_{jr} results

$$E[q_{jr}(v)]=\frac{1}{\sqrt{2\pi}\sigma_v}\frac{2Mg}{L\rho A}\int_{-\infty}^{\infty}sin\left(j\Delta t\frac{\pi v}{L}r\right)e^{-\frac{(v-\mu_v)^2}{2\sigma_v^2}}dv \tag{29}$$

and is given in closed form

$$\mu_{q_{jr}} = E[q_{jr}(v)] = \frac{2Mg}{L\rho A} sin\left(j\Delta t \frac{\pi\mu_v}{L} r \right) e^{\frac{-j^2\Delta t^2\pi^2\sigma_v^2}{2L^2} r^2} \quad (30)$$

The variance matrix $\mathbf{x_j}$ is given as

$$E[\mathbf{p_{jk}}\mathbf{p_{jk}}^T] = \mathbf{Q}_{jk} E[\mathbf{q_j}(v)\mathbf{q_j}(v)^T]\mathbf{Q_{jk}}^T = \mathbf{Q_{jk}}\mathbf{\Sigma_{q_j}}\mathbf{Q_{jk}}^T \quad (31)$$

where the covariance matrix $\mathbf{\Sigma}_{\mathbf{q}_j}$ is defined as

$$\mathbf{\Sigma_{q_{jk}}} = \begin{bmatrix} E[q_{j1}(v)^2] & E[q_{j1}(v)q_{j2}(v)] & \cdots & E[q_{j1}(v)q_{jm}(v)] \\ \vdots & E[q_{j2}(v)^2] & \cdots & E[q_{j2}(v)q_{jm}(v)] \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \cdots & \cdots & E[q_{jm}(v)^2] \end{bmatrix} \quad (32)$$

In order to calculate the variance and the covariance of $\mathbf{p_{jk}}$, it is necessary to calculate first the variance and covariance of $\mathbf{q_j}(v)$.

The covariance of $\mathbf{q_j}(v)$ is expressed as

$$\sigma_{q_{jr}q_{js}} = E[q_{jr}(v)q_{js}(v)] - E[q_{jr}(v)]E[q_{js}(v)] = \int_{-\infty}^{\infty} q_{jr}(v)q_{js}(v)p_v(v)dv - \mu_{q_{jr}}\mu_{q_{js}} \quad (34)$$

where $E[q_{jr}(v)q_{js}(v)]$ results

$$E[q_{jr}(v)q_{js}(v)] = \frac{1}{\sqrt{2\pi}\sigma_v}\left(\frac{2Mg}{L\rho A}\right)^2 \int_{-\infty}^{\infty} sin\left(j\Delta t\frac{\pi v}{L} r\right) sin\left(j\Delta t\frac{\pi v}{L} s\right) e^{-\frac{(v-\mu_v)^2}{2\sigma_v^2}} dv \quad (35)$$

and in closed form

$$E[q_{jr}(v)q_{js}(v)] = \frac{1}{\sqrt{2\pi}\sigma_v}\left(\frac{2Mg}{L\rho A}\right)^2 \frac{1}{2}\cos\left(\frac{j\pi(r-s)\Delta t\mu_v}{L}\right) e^{\frac{-j^2\Delta t^2\pi^2\sigma_v^2(r-s)^2}{2L^2}}$$
$$-\frac{1}{\sqrt{2\pi}\sigma_v}\left(\frac{2Mg}{L\rho A}\right)^2 \frac{1}{2}\cos\left(\frac{j\pi(r+s)\Delta t\mu_v}{L}\right) e^{\frac{-j^2\Delta t^2\pi^2\sigma_v^2(r+s)^2}{2L^2}} \quad (36)$$

In virtue of Eq. (7) the discretization vector $u(x,t)$ may be written as

$$\mathbf{u}(x)^T = \begin{bmatrix} u(x,\Delta t) & u(x,2\Delta t) & \rightleftharpoons & u(x,n\Delta t) \end{bmatrix} \quad (37)$$

Then, considering only the first mode, for shake of simplicity, $\mathbf{u}(x)$ is calculated as

$$\mathbf{u}(x) = \psi_1(x)\mathbf{y}_1 \quad (37)$$

where

$$E[y_1(t)] = E[z_{11}(t)] = \sum_{k=1}^{n} \phi_{1\,1k} E[\mathbf{p_{1k}}] = \sum_{k=1}^{n} \phi_{1\,1k} \mathbf{Q_{1k}} E[\mathbf{q_1}(v)] \tag{38}$$

The mean of $u(x,t)$ denoted as $E[\mathbf{u}]$ results

$$E[\mathbf{u}(x)] = \sin\left(\frac{\pi x}{L}\right) \sum_{k=1}^{n} \phi_{1\,1k} \mathbf{Q_{1k}} E[\mathbf{q_1}(v)] \tag{39}$$

where $\mathbf{u}(x)$ is a matrix that contains the n time steps and the m discrete steps of x. And the root mean square matrix of u(x, t) is calculated as

$$E[\mathbf{u}(x)\,\mathbf{u}(x)^T] = \sin^2\left(\frac{\pi x}{L}\right) \sum_{k=1}^{n} \phi_{1\,1k} \mathbf{Q_{1k}} \boldsymbol{\Sigma}_{\mathbf{q_1}} \mathbf{Q_{1k}}^T \tag{40}$$

5 RESULTS AND CONCLUSIONS

The mean and the variance of the vertical displacement have been calculated using Monte Carlo simulation and by means of the presented methodology.

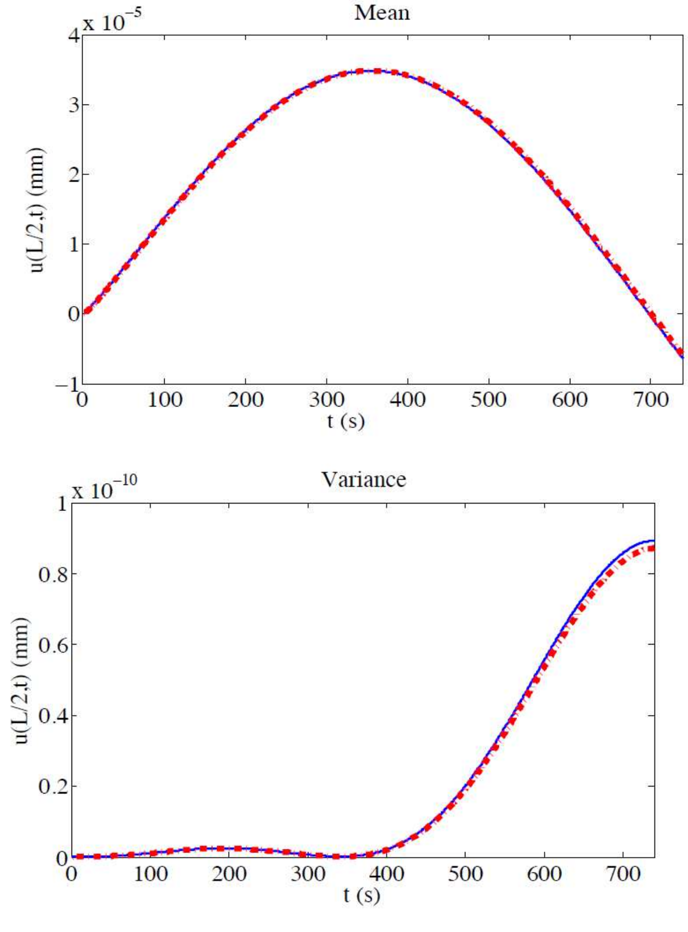

Figure 2 - Mean and variance of the vertical displacement at *L/2* for 1000 samples.

In the Figure 2 are represented the mean and variance of the vertical displacement $u(x,t)$, in blue the presented methodology and in red Monte Carlo simulations for 1000 samples. This example is calculated at x = L/2 for the values: E = 2 10^{11}Pa, I = 10^{-5} m^4, M = 1000 kg, A = 0.01 m^2, L = 1000m, ρ= 4000 kg/m^3, $E_\beta = 10^6 Pas^\beta$, β=0.1, μ_v = 15.3 m/s and σ_v=1.3 m/s.

It has been analysed the influence of stochastic train speed variation in the vertical displacement of fractional viscoelastic rail track ballast.

The mean and the variance of the vertical displacement have been calculated by the presented methodology and compared by means of Monte Carlo simulations.

In order to make the present model and procedure more interesting, and with wider application, the following steps are proposed:

- study the velocity as an stochastic process
- addition of other layers in the rail track
- addition of the sleepers as beam discontinuities

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