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# SOME OBSERVATIONS ON THE APPROXIMATIONS OF THE WIENER PATH INTEGRAL TECHNIQUE

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**Abstract.** *The recently developed approximate Wiener path integral (WPI) technique for determining the stochastic response of nonlinear/hysteretic multi-degree-of-freedom (MDOF) systems has demonstrated a relatively high degree of accuracy. Nevertheless, in the standard implementation of the WPI technique, only the "most probable path" (from the space of all possible paths) contributes to the evaluation of the functional integral for determining the system response transition probability density function (PDF). Clearly, this implies a significant degree of approximation that needs to be quantified. Also, it is shown herein that for a certain class of systems described by stochastic differential equations (SDEs), the WPI approximate solution coincides, notably, with the exact solution. Motivated by the above observations, some preliminary resultsare presented herein pertaining to the accuracy of the WPI approximate technique for a particular class of SDEswith constant diffusion, but nonlinear drift coefficients. Specifically, a bound is derived for the WPI based response transition PDF which can be used as an a priori estimate of the anticipated accuracy of the WPI technique. Further, due to its analytical nature, the bound can be directly used, perhaps, as an approximation of the solution process PDF without resorting to further numerical treatment of the problem.*

## 1 INTRODUCTION

In the field of stochastic dynamics, Monte Carlo simulation methods have been among the most versatile ones for solving stochastic differential equations (SDEs) of general form [1].Nevertheless, in many cases they can be computationally prohibitive; and thus, there is a need for developing alternative approximate analytical/numerical solution techniques such as the ones based on path integrals.The path integral concept was initially introduced by Wiener [2] as a tool for solving problems involving Brownian motion, and was reinvented by Feynman [3] providing a reformulation of quantum mechanics. In general, the SDE solution joint transition probability density function (PDF) can be expressed as a Wiener path integral (WPI), or in other words, asa functional integral over the space of all possible paths. Note, however, that analytical evaluation of the WPI is a highly difficult task in the general case. To circumvent this challenge, research efforts in the literature have focused on applying an extremum condition [4] and accounting, essentially, for the contribution of only one path in the WPI, the so-called most probable path. Of course, it is possible to include additional terms in the related expansion and account for fluctuations around the most probablepath[5], at the expense, however, of computational efficiency.

Further, despite the seemingly significant approximations involved in the above procedure, the accuracy degree demonstrated in several engineering mechanics/dynamics applications is surprisingly high [6-7]. In some cases, as demonstrated herein, the WPI approximate solution coincides, notably, with the exact solution. Motivated by the above observations, some preliminary results are presented herein pertaining to the accuracy of the WPI approximate technique. Specifically, for a particular class of SDEs a bound is derived that can be used as an a priori estimate of the expected accuracy obtained by applying the WPI approximate methodology. Further, due to its analytical nature, in many cases, it can be directly used as an approximation of the solution process PDF without resorting to further numerical treatment of the problem.

## 2 WIENER PATH INTEGRAL IN ENGINEERING MECHANICS

### 2.1 Overview

In general, the transition PDF $p\left(a_f, t_f | a_i, t_i\right)$ of an arbitrary stochastic process $a(t)$ from a point in state space $a_i$ at time $t_i$ to a point $a_f$ at time $t_f$ where $t_f > t_i$, can be expressed as a functional integral over the space of all possible paths $C\{a_i, t_i; a_f, t_f\}$ in the form

$$p\left(a_f, t_f | a_i, t_i\right) = \int_{\{a_i, t_i\}}^{\{a_f, t_f\}} W[a(t)][da(t)]. \tag{1}$$

The WPI of Eq.(1) possesses a probability distribution on the path space as its integrand, which is denoted by $W[a(t)]$ and is called probability density functional. Note that for relatively simple cases, an explicit form of $W[a(t)]$ can be determined. For instance, the probability density functional for the white noise process $v(t)$, i.e.,$E\left(v(t)\right) = 0$ and $E\left(v(t_1)v(t_2)\right) = 2\pi S_0 \delta(t_1 - t_2)$, is given by [8]

$$W[v(t)] = \Phi exp\left[-\int_{t_i}^{t_f} \frac{1}{2} \frac{v(t)^2}{2\pi S_0} dt\right], \tag{2}$$

where$\Phi$ is a normalization coefficient. However, even if the probability density functional is constructed, the analytical solution of the WPI of Eq. (1) is, in general, intractable. Thus, to circumvent the aforementioned challenge, several research efforts have focused on developing approximate techniques for determining the transition PDF. Specifically, in the engineering dynamics field an approximate WPI technique has been developed recently for determining the response transition PDF of multi-degree-of freedom (MDOF) structural systems subject to Gaussian white noise excitations. The technique can account for a wide range of nonlinearity kinds as well as for systems endowed with fractional derivative terms (e.g. [6-7]). In this regard, denoting the response displacement and velocity vectors as $\boldsymbol{w}$ and $\dot{\boldsymbol{w}}$,respectively, the response transition PDF $p\left(\boldsymbol{w_f}, \dot{\boldsymbol{w}}_{\boldsymbol{f}}, t_f | \boldsymbol{w_i}, \dot{\boldsymbol{w}}_{\boldsymbol{i}}, t_i\right)$is given by

$$p\left(\boldsymbol{w_f}, \dot{\boldsymbol{w}}_{\boldsymbol{f}}, t_f | \boldsymbol{w_i}, \dot{\boldsymbol{w}}_{\boldsymbol{i}}, t_i\right) = \int_{\{\boldsymbol{w_i}, \dot{\boldsymbol{w}}_{\boldsymbol{i}}, t_i\}}^{\{\boldsymbol{w_f}, \dot{\boldsymbol{w}}_{\boldsymbol{f}}, t_f\}} \Phi exp\left(-\int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{w}, \dot{\boldsymbol{w}}, \ddot{\boldsymbol{w}}) dt\right) [d\boldsymbol{w}(t)]\,. \tag{3}$$

The right-hand side of Eq.(3) represents a functional integral over the space of all possible paths $C\{\boldsymbol{w_i}, \dot{\boldsymbol{w}}_{\boldsymbol{i}}, t_i; \boldsymbol{w_f}, \dot{\boldsymbol{w}}_{\boldsymbol{f}}, t_f\}$, and $\mathcal{L}(\boldsymbol{w}, \dot{\boldsymbol{w}}, \ddot{\boldsymbol{w}})$ is the Lagrangian function corresponding to structural system under consideration; see [6-7]and references therein for more details. As mentioned earlier, it can be readily seen that the analytical solution of the WPI of Eq.(3) is at least a rather daunting, if not impossible, procedure; thus, an approximate solution is needed. To this aim, it is noted that the largest contribution to the WPI comes from the trajectory for which the integral in the exponential of Eq.(3) becomes as small as possible. Variational calculus rules [4] dictate that this trajectory with fixed end points satisfies the extremality condition

$$\delta \int_{t_i}^{t_f} \mathcal{L}\big((\boldsymbol{w_c}, \dot{\boldsymbol{w}}_{\boldsymbol{c}}, \ddot{\boldsymbol{w}}_{\boldsymbol{c}})\big) dt = 0 \ , \tag{4}$$

where$\boldsymbol{w_c}(t)$ denotes the *"most probable path"* to be determined by the functional optimization problem

$$Min(Max) \qquad J[\boldsymbol{w_c}(t)] = \int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{w_c}, \dot{\boldsymbol{w}}_{\boldsymbol{c}}, \ddot{\boldsymbol{w}}_{\boldsymbol{c}}) dt, \tag{5}$$

together with the boundary conditions $\boldsymbol{w_c}(t_i) = \boldsymbol{w_i}$, $\dot{\boldsymbol{w}}_{\boldsymbol{c}}(t_i) = \dot{\boldsymbol{w}}_{\boldsymbol{i}}$, $\boldsymbol{w_c}(t_f) = \boldsymbol{w_f}$, $\dot{\boldsymbol{w}}_{\boldsymbol{f}}(t_f) = \dot{\boldsymbol{w}}_{\boldsymbol{f}}$. Depending on the complexity of the problem, $\boldsymbol{w_c}(t)$ can be determined either by deriving and solving the Euler-Lagrange (E-L) equations associated with Eq.(4) (e.g. [7]), or, alternatively, by treating directly the deterministic boundary value problem (BVP) of Eq.(5) (e.g. [6]). Once $\boldsymbol{w_c}(t)$ is determined, the transition PDF can be approximated by

$$p\big(\boldsymbol{w_f}, \dot{\boldsymbol{w}}_{\boldsymbol{f}}, t_f \big| \boldsymbol{w_i}, \dot{\boldsymbol{w}}_{\boldsymbol{i}}, t_i\big) \approx \Phi \exp\left(-\int_{t_i}^{t_f} \mathcal{L}(\boldsymbol{w_c}, \dot{\boldsymbol{w}}_{\boldsymbol{c}}, \ddot{\boldsymbol{w}}_{\boldsymbol{c}}) dt\right). \tag{6}$$

Comparing Eqs.(3) and (6), it is seen that only the largest contribution to the WPI of Eq.(3) is considered in the approximation of Eq.(6); this comes from the most probable path $\boldsymbol{w_c}(t)$for which the integral in Eq.(5) becomes as small as possible. It is noted that the approximation of Eq.(6) has demonstrated satisfactory accuracy when compared to pertinent brute-force MCS data for the considered engineering dynamical systems (e.g. [6-7]). Also, notably, in the following example the WPI approximate solution coincides with the exact solution.

## 2.2 Motivation: The stochastic beam bending problem – An exact solution case

Consider a statically determinate Euler-Bernoulli beam satisfying the differential equation

$$\frac{d^2}{dx^2}\left(E(x) I \frac{d^2 w}{dx^2}\right) = l(x), \tag{7}$$

where $x$ denotes the spatial variable, $E(x)$ represents the Young's modulus modeled as a stochastic field; $I$ is the constant cross-sectional moment of inertia; $w(x)$ represents the deflection of the beam; and $l(x)$ denotes a deterministic distributed force. Further, note that Eq. (7) can be integrated twice to produce the internal force (bending moment) which is deterministic and twice more to produce the deflection which is stochastic. In particular, for a given length $L$ of the beam Eq.(7) can be integrated under the boundary conditions $-E(x) I \frac{d^2 w}{dx^2} = M_0$ at $x = 0$ and $-E(x) I \frac{d^2 w}{dx^2} = M_L$ at $x = L$ to obtain

$$-E(x) I \frac{d^2 w}{dx^2} = M(x), \tag{8}$$

where$M(x)$ is the bending moment of the beam. In the following, the inverse of the Young's modulus is assumed to vary randomly along the axis of the beam as

$$\frac{1}{E(x)}=\frac{1}{E_M}(1+v(x)), \tag{9}$$

where$E_0$ is the mean value of the Young's modulus and $v(x)$ represents a homogeneous stochastic field modeled as a white noise process with the properties $E\big(v(x)\big)=0$, and $E\big(v(x_1)v(x_2)\big)=2\pi S_0\delta(x_1-x_2)$, where $S_0$ is the constant white noise power spectrum value. Further, applying the WPI approximate technique [6-7] to the stochastic Eq.(8) the BVP of Eq.(5) becomes

$$Min(Max) \qquad J[w_c(x)]=\int_{x_i}^{x_f}\mathcal{L}\left(x,w_c(x),\frac{dw_c(x)}{dx},\frac{d^2w_c(x)}{dx^2}\right)dx\ , \tag{10}$$

with the boundary conditions $w_c(x_i)=w_{c_i}\,, w_c\big(x_f\big)=w_{c_f}, \dot{w}_c(x_i)=\vartheta_i, \dot{w}_c\big(x_f\big)=\vartheta_f$. Also, the corresponding Euler-Lagrange equation becomes

$$\frac{\partial\mathcal{L}}{\partial w_c}-\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial\dot{w}_c}+\frac{\partial^2}{\partial x^2}\frac{\partial\mathcal{L}}{\partial\ddot{w}_c}=0\ . \tag{11}$$

Next, for the specific case of a cantilever beam with $M(x)=M_0$ the Lagrangian function becomes

$$\mathcal{L}(w_c,\dot{w}_c,\ddot{w}_c)=\frac{1}{2}\frac{\left(\frac{d^2w_c}{dx^2}+\frac{M_0}{E_0I}\right)^2}{2\pi S_0\left(\frac{M_0}{E_0I}\right)^2}. \tag{12}$$

Substituting Eq.(12) into the Euler-Lagrange Eq.(11) and solving yields

$$w_c(x)=c_0+c_1x+c_2x^2+c_3x^3. \tag{13}$$

Applying the boundary conditions $w_c(0)=0$, $\dot{w}_c(0)=0$, $w_c\big(x_f\big)=w_f$, $\dot{w}_c\big(x_f\big)=\vartheta_f$ the coefficients are determined as

$$c_0=0,\ \ c_1=0,\ \ c_2=-\frac{x_f\vartheta_f-3w_f}{{x_f}^2},\ \ c_3=-\frac{-x_f\vartheta_f+2w_f}{{x_f}^3}. \tag{14}$$

Substituting next Eqs.(13-14) into Eq.(6)

$$p\big(w_f,\vartheta_f,x_f|w_i,\vartheta_i,x_i\big)\approx\Phi\exp\left(-\int_{x_i}^{x_f}\mathcal{L}\left(x,w_c(x),\frac{dw_c(x)}{dx},\frac{d^2w_c(x)}{dx^2}\right)dx\right) \tag{15}$$

and manipulating yields the bivariate Gaussian response PDF

$$p\big(w_f,\vartheta_f,x_f|0{,}0{,}0\big)=(2\pi)^{-1}|\boldsymbol{\Sigma}|^{-1/2}\exp\left[-\frac{1}{2}(\boldsymbol{X}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{X}-\boldsymbol{\mu})\right], \tag{16}$$

where$\boldsymbol{X}=(w_f\vartheta_f)^T$ and

$$\boldsymbol{\mu}=-\frac{M_0}{E_0I}\begin{bmatrix}\frac{1}{2}{x_f}^2\\ x_f\end{bmatrix},\boldsymbol{\Sigma}=2\pi S_0\left(\frac{M_0}{E_0I}\right)^2\begin{bmatrix}\frac{1}{3}{x_f}^3 & \frac{1}{2}{x_f}^2\\ \frac{1}{2}{x_f}^2 & x_f\end{bmatrix}. \tag{17}$$

Further, following integration, the marginal PDFs $p\big(w_f,x_f|0{,}0\big)$ and $p\big(\vartheta_f,x_f|0{,}0\big)$ are given, respectively, by

$$p\big(w_f,x_f|0{,}0\big)=\frac{1}{\sqrt{\frac{2}{3}\pi S_0x_f^3\left(\frac{M_0}{E_0I}\right)^2}\sqrt{2\pi}}exp\left(-\frac{1}{2}\frac{(w_f+\frac{M_0}{2E_0I}x_f^2)^2}{\frac{2}{3}\pi S_0x_f^3\left(\frac{M_0}{E_0I}\right)^2}\right) \tag{18}$$

and

$$p\left(\vartheta_f, x_f|0,0\right) = \frac{1}{\sqrt{2\pi S_0 x_f \left(\frac{M_0}{E_0 I}\right)^2}\sqrt{2\pi}} exp\left(-\frac{1}{2}\frac{(\vartheta_f + \frac{M_0}{E_0 I}x_f)^2}{2\pi S_0 x_f \left(\frac{M_0}{E_0 I}\right)^2}\right). \tag{19}$$

Notably, for the specific stochastic beam bending example considered herein, the WPI based analytical closed-form expression of Eq.(16) for the joint response PDF is also the exact one. This can be readily verified by casting the governing Eq.(8) into a standard SDE form [1, 9]

$$\frac{d\vartheta}{dx} = -\frac{M_0}{E_0 I} - \frac{M_0}{E_0 I}\sqrt{2\pi S_0}\eta(x), \tag{20}$$

whose solution is clearly given by Eq.(19) ($\eta(x)$ is a white noise process with unit intensity). Thus, in this case the exact solution of the joint response PDF coincides with the WPI solution that is based on approximating Eq.(3) with Eq.(6). This interesting and encouraging result regarding the accuracy of the WPI approximate technique motivates further research regarding the conditions under which the WPI approximation provides with the exact solution, or more generally, deriving error bounds and determining accuracy estimates for specific classes of governing stochastic dynamics equations. In the following section, some observations and preliminary results towards this aim are presented.

## 3 MATHEMATICAL ASPECTS

### 3.1 A class of SDEs with constant diffusion and nonlinear drift coefficients

Consider the general class of SDEs of the form

$$dY_t = \mu(Y_t)dt + \sigma dB_t, \tag{21}$$

where $B_t$ is a standard Brownian motion, $\sigma$ is a constant, $\mu(\cdot)$denotes a real-valued function and $Y_t$ is the response process to be determined. Note that the stochastic beam bending Eq.(20) is a special case of the above class. Further, it is assumed in the ensuing analysis that standard conditions guaranteeing the existence and the uniqueness of the solution $X_t$ are satisfied [9]. Next, seeking a solution of the form $Y_t = f(t, B_t)$ for Eq.(21), and considering Itô's Lemma[1, 9], i.e.,

$$df(t, B_t) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \frac{\partial f}{\partial x}dB_t\,, \tag{22}$$

yields the following system of equations to be solved for $f(t, B_t)$, i.e.,

$$\frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x) = \mu\big(f(t,x)\big), \qquad \frac{\partial f}{\partial x}(t,x) = \sigma\,. \tag{23}$$

Taking into account that$\sigma$ is constant, Eq.(23) becomes

$$\frac{\partial f}{\partial t}(t,x) = \mu\big(f(t,x)\big), \qquad \frac{\partial f}{\partial x}(t,x) = \sigma\,. \tag{24}$$

Thus, the exact solution, i.e., a process$X_t = f(t, B_t)$, with$X_t = Y_t$a.s. (almost surely) is determined by solving equation$\frac{\partial f}{\partial t} = \mu(f)$, in conjunction with $f(t,x) = \sigma x + c(t)$, where $c(t)$ is a time-dependent function to be evaluated. The latter expression is determined by solving the equation $\frac{\partial f}{\partial x}(t,x) = \sigma$ in Eq.(24). Overall,a solution process of the form$Y_t = \sigma B_t + c(t)$ is provided, which is, clearly, distributed as a Gaussian PDF with mean $c(t)$ and standard deviation $\sigma$.Obviously, for the case where the drift coefficient is constant, i.e., $\mu(Y_t) = \mu$, such as in the case of the bending beam Eq.(20) examined in the section 2, the solution process takes the form$Y_t = \sigma B_t + \mu t$.

Further, applying the WPI approximate solution methodology to Eq.(21) yields aLagrangian function of the form

$$L(y, \dot{y}) = \frac{1}{2}\left[\frac{\dot{y} - \mu(y)}{\sigma}\right]^2, \quad (25)$$

whereas the functional minimization problem of Eq.(5) leads to the E-L equation [6-7]

$$\frac{\partial \mathcal{L}}{\partial y_c} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{y}_c} = 0\,, \quad (26)$$

with the boundary conditions $y_c(t_i) = y_{c_i}\,, y_c(t_f) = y_{c_f}$. Taking into account Eqs.(25-26) yields

$$\ddot{y_c} = \mu(y_c)\frac{\partial \mu(y_c)}{\partial y_c}, \quad (27)$$

which can be transformed into

$$2\dot{y}_c\ddot{y_c} = 2\mu(y_c)\frac{\partial \mu(y_c)}{\partial y_c}\dot{y}_c, \quad (28)$$

or equivalently,

$$\frac{\partial}{\partial t}\dot{y}_c{}^2 = \frac{\partial}{\partial t}\mu(y_c)^2, \quad (29)$$

Eq.(29) leads to

$$\dot{y}_c{}^2 = \mu(y_c)^2 + d\,, \quad (30)$$

where$d$ is a constant. At this point, it is interesting to note the similarity of the derived E-L Eq.(30) to be solved for the most probable path$y_c$, and the form of Eq.(24), i.e., $\frac{\partial f}{\partial t} = \mu(f)$ to be solved for the process$Y_t = f(t, B_t)$. Considering next Eq.(25) and substituting Eq.(30) yields

$$L(y_c, \dot{y}_c) = \frac{1}{2}\left[\frac{2\dot{y}_c{}^2 - d - 2\dot{y}_c\mu(y_c)}{\sigma^2}\right], \quad (31)$$

whereas the integral of Eq.(31) is given by

$$\int_{t_i}^{t_f} L(y_c, \dot{y}_c)dt = \frac{1}{2}\left[\frac{2\int_{t_i}^{t_f}\dot{y}_c{}^2 dt - d(t_f - t_i) - 2M\left(y_{c_f}\right) + 2M\left(y_{c_i}\right)}{\sigma^2}\right], \quad (32)$$

where$M(\cdot)$denotes an antiderivative of $\mu(\cdot)$.Next, utilizing the Cauchy–Schwarz inequality the quantity $2\int_{t_i}^{t_f}\dot{y}_c{}^2 dt$ can be bounded by

$$2\int_{t_i}^{t_f}\dot{y}_c{}^2 dt \geq \int_{t_i}^{t_f}\dot{y}_c{}^2 dt \geq \frac{\left(y_{c_f} - y_{c_i}\right)^2}{t_f - t_i}\,. \quad (33)$$

Combining Eq.(32) and (33) yields

$$\int_{t_i}^{t_f} L(y_c, \dot{y}_c)dt \geq \frac{1}{2\sigma^2}\left[\frac{\left(y_{c_f} - y_{c_i}\right)^2}{t_f - t_i} - d(t_f - t_i) - 2M\left(y_{c_f}\right) + 2M\left(y_{c_i}\right)\right]. \quad (34)$$

Thus, it can be readily seen that a bound for the transition PDF can be given by

$$p\left(y_{c_f}, t_f \middle| y_{c_i}, t_i\right) = F\exp\left(-G\left(y_{c_f}, t_f \middle| y_{c_i}, t_i\right)\right), \quad (35)$$

where

$$G\left(y_{c_f}, t_f \middle| y_{c_i}, t_i\right) = \left[\frac{\left(y_{c_f} - y_{c_i}\right)^2 + \left(-2M\left(y_{c_f}\right) + 2M\left(y_{c_i}\right)\right)\left(t_f - t_i\right)}{2\left(t_f - t_i\right)\sigma^2}\right], \quad (36)$$

and the arbitrary term $\exp\left(\frac{-d\left(t_f - t_i\right)}{2\sigma^2}\right)$has been included in the constant $F$ in Eq.(35). Clearly, the normalization constant $F$ is determined as $F = \frac{1}{\int_{-\infty}^{\infty} \exp\left(-G\left(x, t_f \middle| y_{c_i}, t_i\right)\right) dx}$. It can be readily seen that Eq.(35) can be used as an a priori estimate of the expected accuracy obtained by applying the WPI approximate methodology. Further, in many cases, it can be directly used as an approximation of the solution process PDF obtained at zero computational cost, without resorting to the numerical solution of the E-L Eq.(26).

### 3.2 Numerical example

Consider next the SDE of Eq.(21) with $\mu(y) = -y - \varepsilon y^3$, where $\varepsilon$ is a constantparameter accounting for the degree of nonlinearity. That is,

$$dY_t = (-Y_t - \varepsilon Y_t^3)dt + \sigma dB_t. \quad (37)$$

In the following, the condition $Y(t = 0) = 0$ is considered, and $\sigma = 1$. Note thatfor the SDE of Eq.(37) an antiderivative$M(y)$ is given by $M(y) = -\frac{y^2}{2} - \frac{\varepsilon}{4} y^4$, and thus, Eq.(36) becomes

$$G\left(y_{c_f}, t_f \middle| 0,0\right) = \left[\frac{{y_{c_f}}^2 + \left(-2M\left(y_{c_f}\right) + 2M(0)\right)t_f}{2t_f}\right] = \left[\frac{{y_{c_f}}^2 + (y_{c_f}^2 + \frac{\varepsilon}{2} {y_{c_f}}^4)t_f}{2t_f}\right]. \quad (38)$$

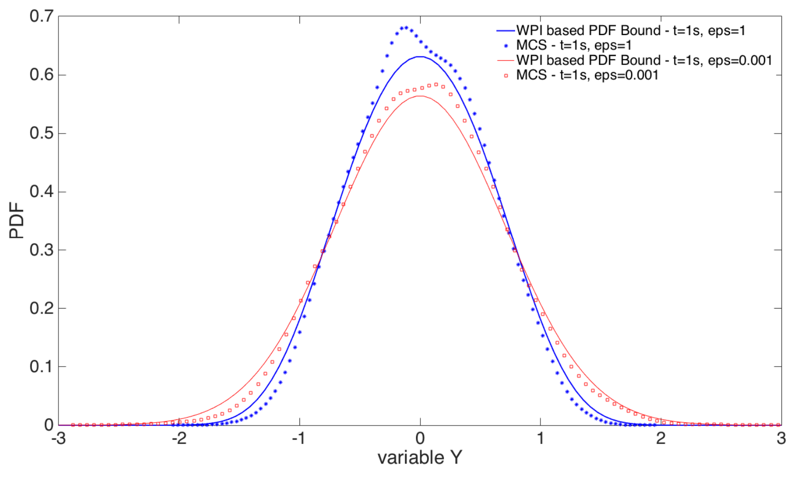


*Figure 1: Wiener path integral based PDF bound for various values of $\varepsilon$; comparisons with MCS data for$t = 1s$*

In Fig.1, the WPI based PDF bound of Eq.(38) is plotted and compared against MCS data (5000 realizations). It is argued that given the accuracy exhibited, the bound can be used directly, perhaps, also as an approximation of the solution process PDF.

## 4 CONCLUDING REMARKS

In this paper, it was shown that for the stochastic beam bending problem the WPI approximate solution coincides, notably, with the exact solution. Motivated by this result, some preliminary work was presented herein pertaining to the accuracy of the WPI approximate technique for a particular class of SDEs. Specifically, a bound for the WPI based response transition PDF has been derived based on the Cauchy-Schwarz inequality for SDEs with constant diffusion, but nonlinear drift coefficients. This bound can be used as an a priori estimate of the expected accuracy obtained by applying the WPI approximate methodology. Further, due to its analytical nature, it can be directly used, perhaps, as an approximation of the solution process PDF without resorting to further numerical treatment of the problem.

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