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HILBERT TRANSFORM BASED STOCHASTIC AVERAGING OF NONLINEAR OSCILLATORS

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Abstract.*Anovel stochastic averaging technique based on a Hilbert transform definition of theoscillator response displacement amplitude is developed. Specifically, a critical step in the conventional stochastic averaging treatment involves the selection of an appropriate period of oscillation over which temporal averaging can be performed. Clearly, for oscillators with nonlinear stiffness defining such a period is not an obvious task.To this aim, an intermediate step is often introduced relating to the linearization of the nonlinear stiffness element, i.e., treating it as response amplitude dependent. Obviously, this additional approximation can potentially decrease the overall accuracy of the technique. Thus, to circumvent some of the above limitations an alternative definition of the amplitude process is consideredherein based on the Hilbert transform. In comparison to a standard definition of the response displacement amplitude, the herein utilized amplitude definition does not require the "a priori" selection of an equivalent natural frequency. Notably, this feature provides with enhanced flexibility in the ensuing stochastic averaging treatment, and can potentially result in higher accuracy. A Duffing oscillator is considered in a numerical example, whereas the derived closed-form analytical expression for the response amplitude stationary probability density function isset vis-a-vis pertinent Monte Carlo simulation data.*

1 INTRODUCTION

Stochastic averaging has been a potent mathematical tool for obtaining approximate solutions to problems involving the vibration response of lightly damped systems to broad-band random excitation [1]. The main features of the technique relate to a Markovian approximation of an appropriately chosen amplitude of the system response, as well as to a dimension reduction of the original $2n$-dimensional problem to an n-dimensional problem. Thus, not only the order of the problem is reduced by half, but also the Markovian character of the response enables the use of well-established techniques for solving the corresponding Fokker-Planck equation and for determining system response statistics [2].

Regarding the choice of the response amplitude, that of theresponse displacement is typically utilized [1, 2], although other alternative choices are available such as that of total energy [3, 4]. Nevertheless, the choice of the standard response displacement amplitudefor systems with nonlinear stiffness elements poses certain limitations in the application of the stochastic averaging technique [1]. A critical step in the stochastic averaging treatment involves the selection of an appropriate period of oscillation over which the temporal averaging can be performed. Clearly, for oscillators with nonlinear stiffness defining such a period is not an obvious task. To this aim, an additional step is often introduced relating to the linearization of the nonlinear stiffness element, i.e., treating it as response amplitude dependent[2, 5]. It can be readily seen that the introduction of this intermediate step can increase the degree of approximation and potentially decrease the overall accuracy as compared to applying the approach to oscillators with nonlinear damping terms but with linear stiffness [1].

In this paper, to circumvent some of the limitations described above an alternative definition of the amplitude process [6] is utilized based on the Hilbert transform [7, 8]. In comparison to a standard amplitude definition of the response displacement, the herein utilized amplitude definition does not require the “a priori” determination of an equivalent natural frequency. Notably, this feature provides enhanced flexibility in the ensuing stochastic averaging treatment, and can potentially result in higher accuracy. A Duffing oscillator is considered in a numerical example, whereas the analytical results are set vis-a-vis pertinent Monte Carlo simulation (MCS) data.

2 MATHEMATICAL FORMULATION

2.1 Conventional stochastic averaging

Consider asingle-degree-of-freedom (SDOF) oscillator with linear damping and a nonlinear stiffness element whose equation is given by

$$\ddot{x} + \beta\dot{x} + g(x) = w(t) \tag{1}$$

where a dot over the variables denotes differentiation with respect to time t; x is the response displacement; $g(x)$ accounts for the nonlinear stiffness element; $\beta = 2\xi_0\omega_0$ is the linear damping coefficient; ω_0 is the natural frequency of the corresponding linear oscillator (i.e. $g(x) = {\omega_0}^2 x$); ξ_0 is the damping ratio; and $w(t)$ represents a Gaussianwhite noise process with a constant power spectrum magnitude S_0.

Next, a critical step in the stochastic averaging treatment involves the selection of an appropriate period of oscillation over which temporal averaging can be performed. Clearly, for oscillators with nonlinear stiffness as in Eq. (1), defining such a period (or equivalently, a natural frequency) is not a straightforward task. In this regard, traditionally, research efforts have focused on combining a statistical linearization treatment with stochastic averaging. Specifically, during the first step of the process an amplitude-dependent equivalent natural frequency $\omega(A)$is defined, and thus, the original nonlinear system of Eq. (1) is approximated by its linearized version

$$\ddot{x} + \beta\dot{x} + \omega^2(A)x = w(t) \tag{2}$$

Once the linearized oscillator is defined, then a standard stochastic averaging treatment [5]can yield a first-order stochastic differential equation (SDE) governing the response amplitude process $A(t)$. Typically, the equivalent stiffness element is determined as the average of $g(x)$ over one cycle of oscillation. That is,

$$\omega^2(A) = \frac{1}{A\pi}\int_0^{2\pi} g[A\cos(\psi)]\cos(\psi)\, d\psi \tag{3}$$

In addition to theoretical difficulties associated with the above handling of cases with nonlinear stiffness [1, 5, 9], it can be readily seen that the introduction of the intermediate step of Eq.(2) (i.e. $g(x) \approx \omega^2(A)x$) increases the degree of approximation, and potentially decreases the overall accuracy degree as compared to a standard stochastic averaging treatment of oscillators with linear stiffness terms [1]. Further, a choice must normally be made regarding the definition of the amplitude process$A(t)$ [6] with

$$A^2(t) = x^2 + \left(\frac{\dot{x}}{\omega(A)}\right)^2 \tag{4}$$

being, perhaps, the most widely utilized. Note that in cases of oscillators with nonlinear damping terms, but with linear stiffness, the intermediate approximation of Eq.(2-3) is not required, and the amplitude of Eq.(4) can be directly defined as $A^2(t) = x^2 + (\dot{x}/\omega_0)^2$; see [1] for a discussion.

2.2 Stochastic averagingbased on a Hilbert transform definition of the amplitude

In this section, to circumvent some of the limitations described in section 2.1 an alternative definition of the amplitude process [6] is utilized based on the Hilbert transform [7, 8]. In particular, relying on the standard assumptions of stochastic averaging [1, 3], and assuming a pseudo-harmonic response behavior, the response processes, x and $\dot{x}$, are defined as

$$x = A_H\, cos\,(\Psi) \tag{5}$$

and

$$\dot{x} = -A_H\dot{\Psi}\ \sin(\Psi) \tag{6}$$

respectively, where A_H is the response amplitude, and Ψ is the response instantaneous phase. BothA_H and $\Psi$are assumed to be slowly varying functions with time. In Eqs.(5-6) the amplitude A_H is defined as

$$A_H{}^2 = x^2 + \tilde{x}^2 \tag{7}$$

while the response instantaneous phase is defined as

$$\Psi = \tan^{-1}\left(\frac{\tilde{x}}{x}\right) \tag{8}$$

and$\tilde{x}$denotes the Hilbert transform of$x$given by

$$\tilde{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{\tau - t}d\tau = \frac{1}{\pi t} * x(t) \tag{9}$$

In Eq.(9), the symbol$*$ denotes the convolution operator. Note that in comparison to the standard amplitude definition of Eq.(4), the Hilbert transform based definition of Eq.(7) does not require the determination of an "equivalent" natural frequency, at least in an "a priori" manner. This feature provides enhanced flexibility in the ensuing stochastic averaging treatment. Further, in addition to the amplitude definition of Eq. (7) that refers to the response displacement process x, an amplitude A_R for the restoring force $g(x)$is defined in the following as

$$g(x) = A_R \, cos \, (\Psi) \tag{10}$$

The restoring force amplitudeA_R is considered to be a slowly varying with time function as well, while it can be readily seen from Eqs.(5) and (10) that in general $A_R = A_R \, (A_H, \Psi)$.The introduction of A_R is motivated by alternative amplitude definitions in the literature such as the energy amplitude [3, 4], and it is suggested that it serves more naturally than A_H to describe the problem at hand where the difficulties in the mathematical treatment relate to $g(x)$. For comparison purposes, note that based on the traditional treatment of section 2.1 and according to the approximation$g(x) \approx \omega^2(A)x$, the amplitude of the restoring force in that case is defined as $\omega^2(A)A$.

Next, differentiating Eq. (7) with respect to time, taking into account the relationships $\tilde{x} = -\dot{x}/\dot{\Psi}$ and $\dot{\tilde{x}} = -\ddot{x}/\dot{\Psi}$, and solving for $\dot{A}_H$yields

$$\dot{A}_H = \frac{\dot{x}}{A\dot{\Psi}^2}\left[w(t) - \beta\dot{x} + x\dot{\Psi}^2 - g(x)\right] \tag{11}$$

Substituting Eqs.(5-6) and Eq.(10) into Eq.(11), yields

$$\dot{A}_H = -\frac{\sin(\Psi)w(t)}{\dot{\Psi}} - \beta A_H \sin^2(\Psi) - \dot{\Psi}A_H \cos(\Psi)\sin(\Psi) - \frac{A_R \; \cos(\Psi)\sin(\Psi)}{\dot{\Psi}} \tag{12}$$

while averaging over one cycle of oscillation (i.e. $\int_0^{2\pi} \dots d\Psi$) leads to

$$\dot{A}_H = -\frac{\beta A_H}{2} - \frac{\sin(\Psi)w(t)}{\dot{\Psi}} \tag{13}$$

Note that in deriving Eq.(13) oscillatory terms of the form $\sin^2(\Psi)$ and $\cos(\Psi)\sin(\Psi)$ have vanished, while the last term on the right-hand-side of Eq. (12) containing the restoring force amplitude disappeared as well. Next, following a similar procedure for Ψ yields the equation

$$\dot{\Psi}^2 = \sin^2(\Psi) - \beta\,\dot{\Psi}\sin(\Psi)\cos(\Psi) + \frac{2}{A_H}\cos(\Psi)\,g(A_H\,cos\,(\Psi)) - \frac{2}{A_H}\cos(\Psi)\,w(t) \tag{14}$$

Applying astraightforward averaging procedure on Eq.(14) the average over one cycle of the term $\cos(\Psi)\,g\big(A_H\,cos\,(\Psi)\big)$ will result to the conventional definition of the equivalent stiffness element $\omega^2(A)$of Eq.(3). However, substituting in Eq.(14) the restoring force amplitude(Eq.(10)), and then averaging over one cycle yields

$$\dot{\Psi}^2 = \frac{A_R}{A_H} - \frac{2}{A}\cos(\Psi)\,w(t). \tag{15}$$

Solving Eq.(15) for $\dot{\Psi}$, and utilizing a Taylor expansion where the first two terms are kept leads to

$$\dot{\Psi} = \sqrt{\frac{A_R}{A_H} - \frac{2}{A_H}\cos(\Psi)\,w(t)} \approx \sqrt{\frac{A_R}{A_H}} - \frac{\cos(\Psi)w(t)}{\sqrt{A_R A_H}}. \tag{16}$$

Next, following a standard stochastic averaging procedure as described in [5], and relying on the broad-band character of the excitation process $w(t)$, Eq. (13) and Eq. (16) can be written as

$$\dot{A}_H = -\frac{\beta A_H}{2} + \frac{\pi S_0}{2\sqrt{A_R A_H}\dot{\Psi}} + \frac{\sqrt{\pi S_0}}{\dot{\Psi}}\bar{\eta}(t), \tag{17}$$

and

$$\dot{\Psi} = \sqrt{\frac{A_R}{A_H}} - \frac{\sqrt{\pi S_0}}{\sqrt{A_R A_H}}\bar{\xi}(t) \tag{18}$$

respectively, where $\bar{\eta}(t)$ and $\bar{\xi}(t)$ are white noise processes with unit intensity.Note that in deriving Eqs.(17-18) the Wiener-Khinchinrelationshiphas been utilized, i.e.,$\int_0^\infty \cos(\dot{\Psi}\tau)\,E[w(t)w(t+\tau)]\,d\tau = \pi S_0$, wherethe derivative of the instantaneous phase

$\dot{\Psi}$ is construed as the frequency (rad/sec).Additional simplifications can be made to Eq.(17) if only the mean value of $\dot{\Psi}$ as given in Eq.(18) (i.e. $\dot{\Psi} = \sqrt{A_R/A_H}$) is substituted in Eq.(17). That is,

$$\dot{A} = -\frac{\beta A}{2} + \frac{\pi S_0}{2A_R} + \frac{\sqrt{\pi S_0}}{\sqrt{\frac{A_R}{A_H}}}\bar{\eta}(t) \tag{19}$$

Related to Eq.(19) is the Fokker-Planck equation [10]

$$-\frac{d}{dA_H}\left[\left(-\frac{\beta A_H}{2} + \frac{\pi S_0}{2A_R}\right)p(A_H,t)\right] + \frac{1}{4}\frac{d}{dA_H}\left\{\frac{\pi S_0}{\frac{A_R}{A_H}}\frac{dp(A_H)}{dA_H} + \frac{d}{dA_H}\left[\frac{\pi S_0}{\frac{A_R}{A_H}}p(A_H,t)\right]\right\} = \frac{\partial p(A_H,t)}{\partial t} \tag{20}$$

which must be solved for the response amplitude PDF $p(A_H,t)$. Note that for the linear case, i.e. $g(x) = \omega_0^2 x$, considering Eqs.(5) and (10) yields $A_R/A_H = \omega_0^2$. In that case the F-P equation admitsthe Rayleigh PDF as the solution for the stationary response amplitude PDF, i.e. $p(A_H) = \frac{A_H}{\sigma^2}\exp\left[-\frac{{A_H}^2}{2\sigma^2}\right]$, where $\sigma^2 = \frac{\pi S_0}{\beta\omega_0^2}$. In the general nonlinear case, clearly, the dependence of A_R on Ψ is to be eliminated (e.g. via an averaging scheme), and an explicit relationship is to be found betweenA_R and A_H of the form $A_R = A_R\,(A_H)$. In this manner Eq.(19) will depend only on A_H and Eq.(20) can be solved for $p(A_H,t)$. The approach is demonstrated in the following section considering a Duffing nonlinear oscillator.

3 NUMERICAL EXAMPLE: DUFFING OSCILLATOR

For the case of a hardening Duffing oscillator, the nonlinear function of Eq.(1) becomes

$$g(x) = \omega_0^2 x(1 + \epsilon x^2), \tag{21}$$

where the parameter$\epsilon > 0$ capturesthe nonlinearity strength. Further, the conventional averaging of Eq.(3) yields an amplitude-dependent equivalent stiffness element $\omega^2(A)$, and a restoring force amplitude of the form

$$\omega^2(A)A = \omega_0^2 A\left[1 + \epsilon\frac{3}{4}A^2\right], \tag{22}$$

whereas considering Eq.(10) a relationship for the Hilbert based restoring force amplitude of the form $A_R = A_R\,(A_H,\Psi)$ is determined. That is,

$$A_R = \omega_0^2 A_H\left[1 + \epsilon {A_H}^2\cos^2(\Psi)\right]. \tag{23}$$

Next, to eliminate the dependence of A_R on Ψ in Eq.(23), a potential treatment relates to applying an averaging scheme over one cycle of oscillation to the nonlinear term $\epsilon {A_H}^2\cos^2(\Psi)$ yielding

$$A_R = \omega_0^2 A_H\left[1 + \epsilon\frac{1}{2}{A_H}^2\right]. \tag{24}$$

An alternative treatment relates to accounting for the maximum influence of the nonlinearity, and thus, considering the envelope of the nonlinear term $\epsilon {A_H}^2\cos^2(\Psi)$. That is,

$$A_R = \omega_0^2 A_H\left[1 + \epsilon {A_H}^2\right]. \tag{25}$$

The accuracy of the three approximations of Eqs.(22) and (24-25) in capturing the restoring force amplitude is examined in the following example. Specifically, considering a Duffing oscillator with the parameter values $\omega_0 = 2\pi$ rad/s, $\xi = 0.01$, $\beta = 2\xi\omega_0$,ϵ=1, and $S_0 = 2\xi\omega_0^3/\pi$, Fig.1 shows a typical realization of the restoring force $g(x)$ together with the three approximations of the restoring force amplitude of Eqs.(22) and (24-25). It can be readily seen that only Eq.(25) can capture with high accuracy the envelope of the restoring force, while the alternative two approximations (Eq.(22) and Eq.(24)) underestimate the peaks of the restoring force significantly.

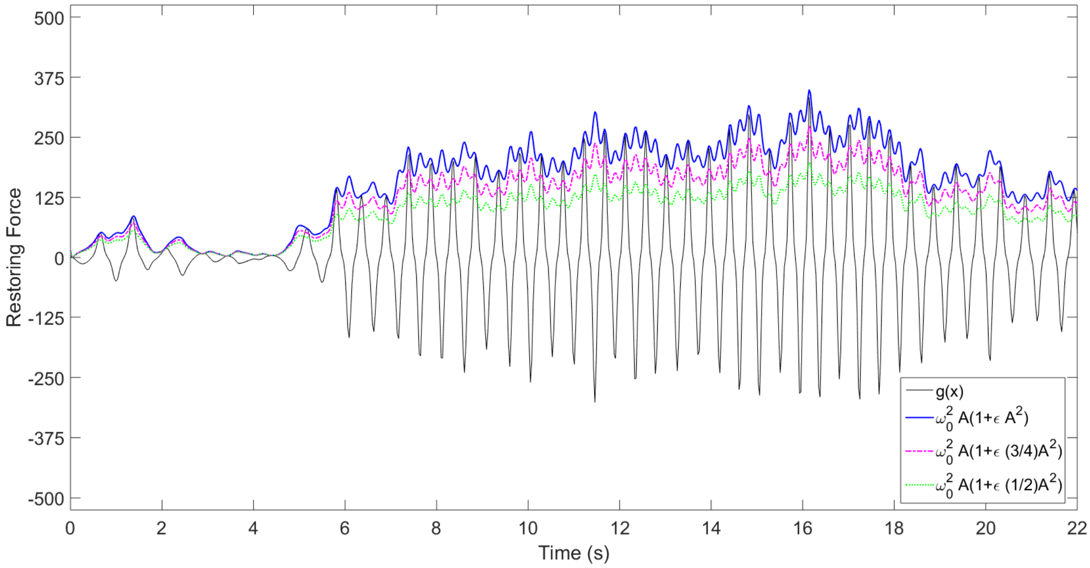

Figure1: Hardening Duffing oscillator restoring force typical time-history, and comparisons between various approximations of the restoring force amplitude.

Thus, it is anticipated that utilizing Eq.(25) in the F-P Eq.(20) will yield a more accurate response amplitude PDF than the one corresponding to the traditional stochastic averaging scheme (Eq.(22)). In particular, considering the standard approximation of Eq.(22), substituting inthe F-P Eq.(20), and solving for the stationary(i.e. $\frac{\partial p(A,t)}{\partial t} = 0$) response amplitude PDF yields [5, 10]

$$p(A) = C\frac{A\sqrt{1+\frac{3}{4}\epsilon A^2}}{\sigma^2}\exp\left[-\frac{\left(\frac{A^2}{2}+\epsilon\frac{3A^4}{16}\right)}{\sigma^2}\right]. \tag{26}$$

where C is a normalization constant.Next, substituting Eq.(25) into the F-P Eq.(20), and solving for the stationary response (Hilbert transform based) amplitude PDF yields

$$p(A_H) = C_H\frac{A_H\sqrt{1+\epsilon A_H{}^2}}{\sigma^2}\exp\left[-\frac{\left(\frac{A_H{}^2}{2}+\epsilon\frac{A_H{}^4}{4}\right)}{\sigma^2}\right]. \tag{27}$$

where $C_H$is a normalization constant.For comparison, the analytical exact expression for the stationary amplitude PDF of a Duffing oscillator derived by Crandall [11] following two distinct pathways (i.e. based on energy considerations, and on peak statistics) is included as well. This is given by

$$p(A_E) = \frac{4\sigma F(k,\pi/2)}{I(\epsilon)\sqrt{1+\epsilon A_E{}^2}}\left(\frac{A_E+\epsilon A_E{}^3}{\sigma^2}\right)\exp\left[-\frac{\left(\frac{A_E{}^2}{2}+\epsilon\frac{A_E{}^4}{4}\right)}{\sigma^2}\right], \tag{28}$$

where $k = \sqrt{\frac{\epsilon A_E}{2+2\epsilon A_E{}^2}}$,$I(\epsilon) = 2\sqrt{2\pi}\int_0^\infty \exp\left[-\left(\frac{A_E{}^2}{2\sigma^2}+\epsilon\frac{A_E{}^4}{4\sigma^2}\right)\right]dA$, and $F(k,\pi/2)$ is the complete elliptic integral of the first kind.

The stationary amplitude PDFs of Eqs.(26-28) are plotted in Fig.2 and compared with

Monte Carlo simulation based response amplitude PDF estimates utilizing 10,000 realizations. To this aim, a standard 4th order Runge-Kuttanumerical integration scheme has been utilized for solving the nonlinear governing Eq.(1) in conjunction with Eq.(21). In producing the MCS based PDF estimates both the Hilbert transform definition of the response amplitude (Eq.(7))was applied on the response displacement realizations, and the solution of the polynomial Eq.(4) was utilized. Focusing on Fig.(2), it is seen that both definitions yield approximately the same MCS based amplitude PDF with very minor differences.

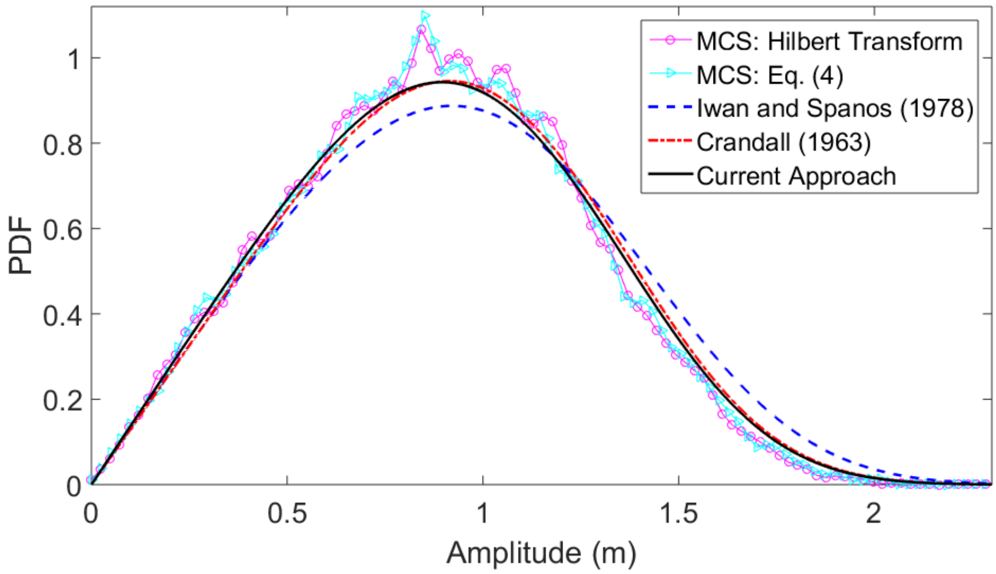

Figure2: Hilbert transform based amplitude stationary PDF, and comparisons with various approximate amplitudePDFsand with pertinent Monte Carlo simulation data.

Regarding the accuracy of the approximate stochastic averaging based amplitude PDFs, it can be readily seen that the Hilbert transform based amplitude PDF derived herein (Eq.(27)) exhibits significantly higher accuracy than the amplitude PDF based on the conventional stochastic averaging treatment. It can perhaps be argued that one of the reasons for this enhanced accuracy is the ability of the herein introduced restoring force amplitude A_R to capture the envelope of the restoring force $g(x)$better than $\omega^2(A)A$. In fact, the Hilbert transform based PDF exhibits practically the same accuracy level as the analytical exact solution of Eq.(28). Note, however, that the analytical nature of the approach byCrandall [11], renders it case-dependent, and clearly, lacks the versatility of a stochastic averaging treatment.

4 CONCLUSION

A novel stochastic averaging technique based on a Hilbert transform definition of the oscillator response displacement amplitude has been proposed. It has been shown that in comparison to a standard amplitude definition of the response displacement, the herein utilized amplitude definition does not require the “a priori” determination of an equivalent natural frequency. In fact, this feature has provided enhanced flexibility in the ensuing analysis, and has enabled the determination of a restoring force amplitude capable of capturing the envelope of the restoring force $g(x)$more accurately than a conventional stochastic averaging treatment of the problem. The overall enhanced accuracy of the technique has been demonstrated via a numerical example considering the Duffing hardening oscillator, and by comparison with pertinent Monte Carlo simulation data. Obviously, the concept of stochastic averaging on Hilbert transform based amplitudes has its own meritand can be supplemented by alternative approximations of relevant quantities not discussed in this paper.

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