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DICAM

ON DUALITY OF DEFINITION OF SAMPLE PATHS DISCONTINUITY IN IMPULSE PROBLEMS: LEFT-CONTINUOUS WITH RIGHT LIMITS (C.ÁG.L.ÁD.) VS. RIGHT-CONTINUOUS WITH LEFT LIMITS (C.ÁD.L.ÁG.)

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Abstract. .*In the present paper two approaches to the problem of Erlang renewal impulse processes are discussed. In the approach used in the theory of stochastic point processes the discontinuous sample paths of the counting process and those of an introduced auxiliary pure-jump stochastic process are assumed to be left-continuous with right limits (c. à g. l. à d. - continue à gauche limite à droite). In the usual, classical approach used in general in mathematics those sample paths are right-continuous with left limits (c. à d. l. à g. - continue à droite limite à gauche). At the example of Erlang renewal impulse processes it is shown that both approaches lead to the same results, but the determination of necessary expectations is more straightforward if c. à g. l. à d. assumption is made.*

1 INTRODUCTION

In problems of non-Poisson (e.g. renewal) impulse processes the state vector of the dynamic system is not a Markov process. The exact conversion of the original non-Markov pulse problem into a Markov one is in some cases possible owing to the introduction of an auxiliary pure-jump stochastic process [1-3]. This must be defined (constructed) in such a way that it selects some impulses from the primitive train driven by a single Poisson process, or by two independent Poisson processes. The values of impulses in such a primitive train are compounded with (multiplied by) the values of the auxiliary pure-jump stochastic process and as a result, only some of the values of impulses in a primitive train remain non-zero, all others being equal to zero. In a definition of the auxiliary pure-jump stochastic process of crucial importance is a proper definition of the discontinuity of its sample paths. In the theory of stochastic point processes the discontinuous sample paths of the counting process are assumed to be left-continuous with right limits (c. à g. l. à d. - continue à gauche limite à droite). It may be shown that such an assumption leads to a straightforward determination of necessary expectations. This is so, because the increments of the counting process are then defined on an anticipating time interval $[t,t+dt)$ and if the counting process is a Poisson process its increment on $[t,t+dt)$ is statistically independent of $N(t)$ and, more importantly it is statistically independent of the state vector at t. However, in mathematics it is usually assumed that the discontinuous functions are at jumps right-continuous with left limits (c. à d.

l. à g. - continue à droite limite à gauche). Both assumptions are analysed and compared. At the example of Erlang renewal impulse processes it is shown that the usual c. à d. l. à g. assumption leads to the same results, but the determination of necessary expectations is more involved.

2 STATEMENT OF A PROBLEM FOR ERLANG RENEWAL IMPULSE PROCESSES

Consider a simple dynamic system under a renewal impulse process excitation

$$\ddot{Y}(t)+2\zeta\omega\dot{Y}(t)+\omega^2 Y(t)=b\sum_{i,R=1}^{R(t)} P_{i,R}\,\delta\left(t-t_{i,R}\right), \tag{1}$$

where the time instants **Error!** are driven by an Erlang renewal process $R(t)$ with integer parameter k, whose events are every kth Poisson events.

The pertinent state vector formulation is

$$d\mathbf{Y}(t)=c\left(\mathbf{Y}(t),t\right)dt+b\left(\mathbf{Y}(t),t\right)P(t)\,dR(t),\quad \mathbf{Y}(0)=y_0, \tag{2}$$

where the term $b\left(\mathbf{Y}(t),t\right)$ (analogue of the diffusion coefficient) may be obtained in some problems by a suitable conversion of the usual differential equation of motion into a stochastic one [4-7]. In some other problems, where the governing stochastic differential equations are formulated in terms of anticipating differentials, this term appears directly [8,9].

Exact conversion of the original non-Markov problem into a Markov one is done by the following recasting of the impulse process (replacement valid with probability 1) [1,2]

$$\sum_{i,R=1}^{R(t)} P_{i,R}\,\delta\left(t-t_{i,R}\right)=\sum_{i=1}^{N(t)}\rho\left(N\left(t_i\right)\right)P_i\,\delta\left(t-t_i\right) \tag{3}$$

where $\rho\left(N\left(t_i\right)\right)$ is an **auxiliary, pure jump, zero-one stochastic process**, which selects every kth impulse from the train driven by a Poisson process $N(t)$, as shown in the Fig. 1.

The jump process $\rho\left(N\left(t_i\right)\right)$ must be defined (constructed) in such a way that it has value 1 only at every kth Poisson event and 0 at all other Poisson events.

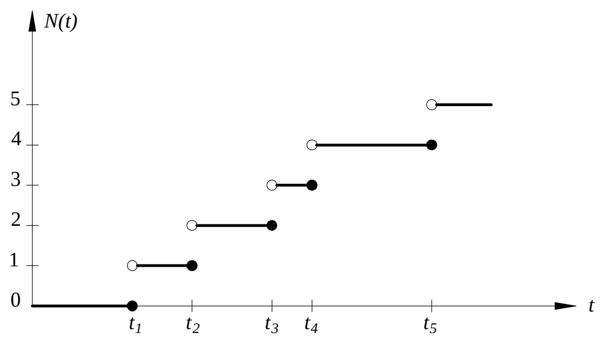

Fig. 1: Train of Erlang impulses: ×: Poisson-driven points, o: Erlang-driven points.

The random counting (stochastic point) process $N(t)$ is defined as a number of counts (points) in the time interval $[0,t)$, i.e. excluding the one that possibly occurs at the time t

[10,11]. Consequently, the sample paths of $N(t)$ are left-continuous with right limits (c. à g. l. à d. - continue à gauche limite à droite). Accordingly, the sample paths of the jump process $\rho\left(N\left(t_i\right)\right)$ are also left-continuous with right limits. However an equivalent definition is also possible, where the sample paths of $N(t)$ and those of $\rho\left(N\left(t_i\right)\right)$ right-continuous with left limits (c. à d. l. à g. - continue à droite limite à gauche).

Validity of the above replacement implies the equivalence of the increments

$$dR(t) = \rho\left(N(t)\right)dN(t) \tag{4}$$

In stochastic analysis (e.g. in the derivation of equations for moments) the expectation $E\left[R(t)\right] = E\left[\rho\left(N(t)\right)dN(t)\right]$ has to be evaluated. It will be shown that the method of evaluation of this expectation is different for two different definitions of the jump process, but leads to the same result.

An Erlang process $R(t)$ is an ordinary renewal process, i.e. its first waiting time (the time elapsed until the first event) has the same probability distribution as all subsequent inter-arrival times. Its so-called ordinary renewal density $h_o(t)$ is defined as [10]

$$\left.\begin{aligned} &\Pr\left\{dR(t)=1\right\} = h_o(t)\,dt \\ &\Pr\left\{dR(t)=0\right\} = 1 - h_o(t)\,dt + o(dt) \end{aligned}\right\} \Rightarrow h_o(t) = E\left[R(t)\right], \tag{5}$$

where the conventionally denoted differential $dR(t)$ is an increment of the renewal process $R(t)$ on an infinitesimal time interval.

3 ERLANG RENEWAL IMPULSE PROCESS WITH INTEGER PARAMETER k=2

The ordinary renewal density of the Erlang process with integer parameter k=2 is equal to

$$h_o(t) = \frac{\nu}{2}\left(1 - \exp(-2\nu t)\right) \tag{6}$$

As the events of an Erlang renewal process with k=2 are every second events of a homogeneous Poisson process with mean rate ν, the increment $dR(t)$ may be expressed in terms of the increment $dN(t)$ of the Poisson process by (4) if the function $\rho(N(t))$ may be found which excludes every second Poisson event. This function, or a jump process, must be a zero-one valued function.

If the sample paths of the counting Poisson process $N(t)$ are assumed to be left-continuous with right limits (c. à g. l. à d. - continue à gauche limite à droite), the event that possibly occurs at the time t is not counted (is excluded), consequently the conventionally denoted differential $dN(t)$ should be regarded as increment $dN(t)=N(t+dt)-N(t)$, of the counting process $N(t)$ on an anticipating infinitesimal time interval $[t,t+dt)$.

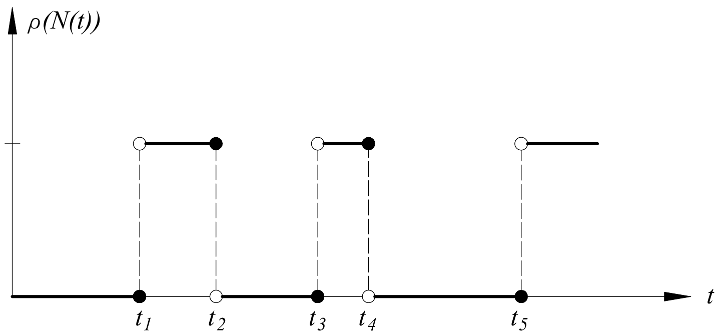

Fig. 2: Left-continuous with right limits (c. à g. l. à d.) sample path of a counting process $N(t)$.

The jump process $\rho(N(t))$ is then defined as

$$\rho\left(N\left(t\right)\right)=\frac{1}{2}\left(1-\left(-1\right)^{N(t)}\right),\; N\left(0\right)=0, \Rightarrow \rho\left(0\right)=\rho\left(N\left(0\right)\right)=0 \tag{7}$$

and

$$dR\left(t\right)=\frac{1}{2}\left(1-\left(-1\right)^{N(t)}\right)dN\left(t\right) \tag{8}$$

The jump process $\rho(N(t))$ is shown in the Fig. 3. Its sample paths are, like those of $N(t)$, left-continuous with right limits.

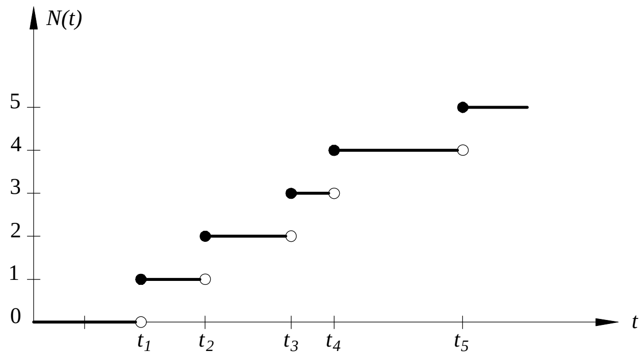

Fig. 3: Left-continuous with right limits (c. à g. l. à d.) sample path of a jump process $\rho(N(t))$.

The renewal density is

$$h_o\left(t\right)dt=E\left[dR\left(t\right)\right]=E\left[\rho\left(N\left(t\right)\right)dN\left(t\right)\right] \tag{9}$$

As the increment $dN(t)$ is defined on the anticipating infinitesimal time interval $[t,t+dt)$, the intervals $[0,t)$ and $[t,t+dt)$ are disjoint hence the increment $dN(t)$ is independent of the value $\rho(N(t))$, thus

$$h_o\left(t\right)dt=E\left[\rho\left(N\left(t\right)\right)dN\left(t\right)\right]=E\left[\frac{1}{2}\left(1-\left(-1\right)^{N(t)}\right)\right]E\left[dN\left(t\right)\right]=\frac{\nu}{2}\left(1-E\left[\left(-1\right)^{N(t)}\right]\right)dt \tag{10}$$

The expectation $E\left[\left(-1\right)^{N(t)}\right]$ is evaluated as follows

$$E\left[(-1)^{N(t)}\right]=\sum_{k=0}^{\infty}(-1)^k\frac{(\nu t)^k}{k!}\exp(-\nu t)=\exp(-\nu t)\sum_{k=0}^{\infty}\frac{(-\nu t)^k}{k!} \\ =\exp(-\nu t)\exp(-\nu t)=\exp(-2\nu t) \tag{11}$$

The result is, indeed

$$h_o(t)=\frac{\nu}{2}\big(1-\exp(-2\nu t)\big) \tag{12}$$

Alternatively, the random counting (stochastic point) process $N(t)$ may be defined as a number of counts (points) in the time interval $(0,t]$, i.e. including the one that possibly occurs at the time t. Accordingly, its sample paths are right-continuous with left limits (c. à d. l. à g. - continue à droite limite à gauche). Consequently the conventionally denoted differential $dN(t)$ should be regarded as increment $dN(t)=N(t)–N(t–dt)$ of the counting processes $N(t)$ on a backward infinitesimal time interval $(t–dt,t]$ and is not statistically independent of $N(t)$.

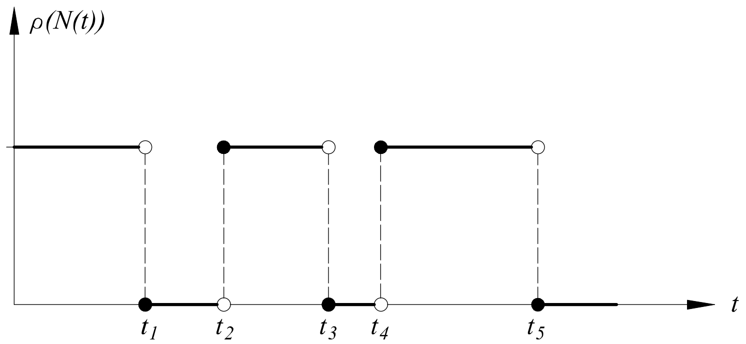

Fig. 4: Right-continuous with left limits (c. à d. l. à g.) sample path of a counting process $N(t)$.

Then the jump process $\rho(N(t))$ is defined as

$$\rho\big(N(t)\big)=\frac{1}{2}\Big(1+(-1)^{N(t)}\Big),\ N(0)=0,\Rightarrow\rho(0)=\rho\big(N(0)\big)=1 \tag{13}$$

The jump process $\rho(N(t))$ is shown in the Fig. 5. Its sample paths are, like those of $N(t)$, right-continuous with left limits.

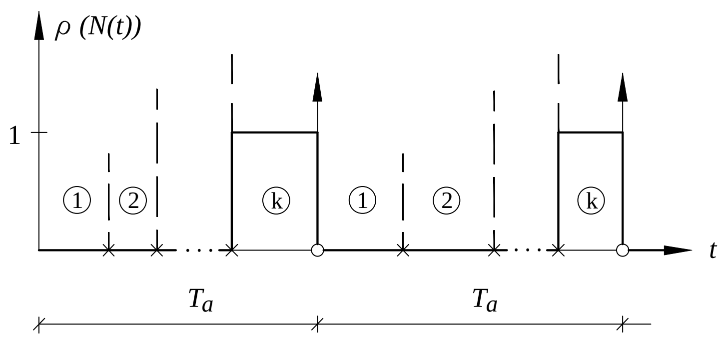

Fig. 5: Right-continuous with left limits (c. à d. l. à g.) sample path of a jump process $\rho(N(t))$.

Now

$$h_o(t)dt = E\left[\rho(N(t))dN(t)\right] = E\left[\frac{1}{2}\left(1+(-1)^{N(t)}\right)dN(t)\right] = \frac{1}{2}\left(\nu dt - E\left[(-1)^{N(t)}dN(t)\right]\right), \quad (14)$$

however as the increments $\rho(N(t-dt,t))$ and $dN(t)=N(t)-N(t-dt)$ are defined on the same (overlapping) time interval, the expectation $E\left[(-1)^{N(t)}dN(t)\right]$ is evaluated in a different way:

$$\begin{aligned}
&E\left[(-1)^{N(t)}dN(t)\right] = E\left[(-1)^{(N(t-dt)+N(t-dt,t))}dN(t)\right] = \\
&E\left[(-1)^{N(t-dt)}\right]E\left[(-1)^{N(t-dt,t)}dN(t)\right] = \\
&\exp(-2\nu(t-dt))E\left[(-1)^{N(t-dt,t)}dN(t)\right] = \exp(-2\nu t)\exp(-2\nu dt)E\left[(-1)^{N(t-dt,t)}dN(
\end{aligned} \quad (15)$$

Further

$$\exp(-2\nu dt) = 1 + 2\nu dt + \ldots \approx 1 \quad (16)$$

and

$$E\left[(-1)^{N(t-dt,t)}dN(t)\right] = (-1)^0 \cdot 0 \cdot (1-\nu dt) + (-1)^1 \cdot 1 \cdot (\nu dt) = -\nu dt \quad (17)$$

Consequently

$$E\left[(-1)^{N(t)}dN(t)\right] = -\exp(-2\nu t)\nu dt \quad (18)$$

and, as before

$$h_o(t) = \frac{\nu}{2}\left(1-\exp(-2\nu t)\right) \quad (19)$$

4 RLANG RENEWAL IMPULSE PROCESS WITH AN ARBITRARY INTEGER PARAMETER k

Under the assumption that the sample paths of the counting Poisson process $N(t)$ are left-continuous with right limits, for an Erlang renewal process with an arbitrary integer parameter k the jump process $\rho(N(t))$ is expressed as [1,2]

$$\rho(N(t)) = \frac{1}{k}\sum_{j=0}^{k-1}\exp\left(i2\pi\frac{j(N(t)+1)}{k}\right) = \frac{1}{k}\sum_{j=0}^{k-1}\exp\left(i2\pi\frac{jN(t)}{k}\right)\exp\left(i2\pi\frac{j}{k}\right) \quad (20)$$

The ordinary renewal density is evaluated from

$$\begin{aligned}
h_o(t)dt &= E\left[\rho(N(t))dN(t)\right] = \frac{1}{k}\sum_{j=0}^{k-1}E\left[\exp\left(i2\pi\frac{jN(t)}{k}\right)\right]\exp\left(i2\pi\frac{j}{k}\right)E\left[dN(t)\right] \\
&= \frac{1}{k}\sum_{j=0}^{k-1}E\left[\exp\left(i2\pi\frac{jN(t)}{k}\right)\right]\exp\left(i2\pi\frac{j}{k}\right)\nu dt
\end{aligned} \quad (21)$$

If the sample paths of the counting Poisson process $N(t)$ are right-continuous with left limits, the jump process $\rho(N(t))$ must be expressed as

$$\rho\left(N(t)\right)=\frac{1}{k}\sum_{j=0}^{k-1}\exp\left(i2\pi\frac{jN(t)}{k}\right) \tag{22}$$

Consequently the ordinary renewal density is evaluated from

$$\begin{aligned} h_o(t)dt &= E\left[R(t)\right]=\left[\rho\left(N(t)\right)dN(t)\right]= \\ &=\frac{1}{k}\sum_{j=0}^{k-1}E\left[\exp\left(i2\pi\frac{jN(t-dt)}{k}\right)\exp\left(i2\pi\frac{jN(t-dt,t)}{k}\right)N(t-dt,t)\right]= \\ &=\frac{1}{k}\sum_{j=0}^{k-1}E\left[\exp\left(i2\pi\frac{jN(t-dt)}{k}\right)\right]E\left[\exp\left(i2\pi\frac{jN(t-dt,t)}{k}\right)N(t-dt,t)\right] \end{aligned} \tag{23}$$

The expectation $E\left[\exp\left(i2\pi\frac{jN(t-dt)}{k}\right)\right]$ is evaluated as

$$E\left[\exp\left(i2\pi\frac{jN(t-dt)}{k}\right)\right]=\sum_{r=0}^{\infty}\exp\left(i2\pi\frac{jr}{k}\right)\frac{\nu^r(t-dt)^r}{r!}\exp\left(-\nu(t-dt)\right) \tag{24}$$

The following evaluation holds if the terms of order $o(dt)$ are neglected

$$(t-dt)^r\exp\left(-\nu(t-dt)\right)=\sum_{s=0}^{r}t^{r-s}(-dt)^s\frac{r!}{(r-s)!s!}\exp(-\nu t)\exp(\nu dt)\approx t^r\exp(-\nu t) \tag{25}$$

Hence

$$E\left[\exp\left(i2\pi\frac{jN(t-dt)}{k}\right)\right]=\sum_{r=0}^{\infty}\exp\left(i2\pi\frac{jr}{k}\right)\frac{(\nu t)^r}{r!}\exp(-\nu t)=E\left[\exp\left(i2\pi\frac{jN(t)}{k}\right)\right] \tag{26}$$

The expectation $E\left[\exp\left(i2\pi\frac{jN(t-dt,t)}{k}\right)N(t-dt,t)\right]$ is evaluated as

$$\begin{aligned} &E\left[\exp\left(i2\pi\frac{jN(t-dt,t)}{k}\right)N(t-dt,t)\right] \\ &=\exp\left(i2\pi\frac{j\cdot 0}{k}\right)\cdot 0\cdot(1-\nu dt)+\exp\left(i2\pi\frac{j\cdot 1}{k}\right)\cdot 1\cdot\nu dt=\exp\left(i2\pi\frac{j\cdot 1}{k}\right)\nu dt \end{aligned} \tag{27}$$

Finally

$$h_o(t)dt=E\left[\rho\left(N(t)\right)dN(t)\right]=\frac{1}{k}\sum_{j=0}^{k-1}E\left[\exp\left(i2\pi\frac{jN(t)}{k}\right)\right]\exp\left(i2\pi\frac{j}{k}\right)\nu dt, \tag{28}$$

as before.

5 CONCLUDING REMARKS

Two definitions of the counting process and its sample paths are discussed. In the first one (customary in the theory of stochastic point processes) the discontinuous sample paths of the counting process and those of an introduced auxiliary pure-jump stochastic process are assumed to be left-continuous with right limits (c. à g. l. à d.). In the second one (a standard one in mathematics) those sample paths are right-continuous with left limits (c. à d. l. à g.). At the example of Erlang renewal impulse processes it is shown that both definitions lead to the same results, but the determination of necessary expectations is easier if sample paths are assumed to be left-continuous with right limits (c. à g. l. à d.).

6 ACKNOWLEDGEMENTS

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