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RECENT DEVELOPMENTS IN SOLUTION OF UNCERTAINTY PROPAGATION PROBLEMS VIA NEUMANN SERIES

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Abstract.*This paper reviews two recent theoretical developments which drastically speed the solution of uncertainty propagation problems using the Neumann series, with application to structural mechanics. The Neumann method consists in approximation the inverse of the systems stiffness matrix by an infinite series. Approximate solutions of converging accuracy are obtained by truncating the series. The first result is based on unexplored properties of the Neumann series, and allows establishing bounds for the realizations of the response process. The second result derives from an error minimization problem, obtained by comparing the true stiffness matrix inverse to a first-order Neumann approximation of the same inverse. A lambda (λ) factor is introduced, and employed to solve the error minimization problem. An analytical solution is obtained, which efficiently yields very accurate results for a first order Neumann approximation of the stiffness matrix inverse. Novel results, exploring synergies between these two solutions, are also described herein. The three novel methods, as well as the two original, are applied in the solution of stochastic plate bending problem, with Winkler and Pastenak types of foundation. These unpublished results confirm that the Neumann-λmethod provides very accurate results, at a fraction of the cost of pure Monte Carlo simulation.*

1 INTRODUCTION

The Monte Carlo simulation method remains a popular, yet computationally expensive tool for analyzing uncertainty propagation problems in mechanics. The computational cost of Monte Carlo simulation can easily become prohibitive, for highly non-linear problems and

complex geometries. More efficient, intrusive methods have recently been developed, such as the stochastic finite element method [1] or stochastic Galerkin Method [2-6]. Intrusive methods have the inconveniency of requiring full re-programming of conventional finite element software. Hence, non-intrusive Monte Carlo simulation methods remain popular in the solution of stochastic mechanics problems.

In linear stochastic mechanics, the numerical solution of a differential equation is replaced by the solution of a linear system of algebraic equations (stiffness matrix). When Monte Carlo simulation is employed, for each system realization, the stiffness matrix needs to be inverted. The Neumann series can be used is to replace the matrix inversions by a truncated series expansion. However, depending on the number of terms in the Neumann series, the number of operations to be performed may become larger than for the actual matrix inversion.

The Neumann series has been employed in the solution of uncertainty propagation problems in mechanics by a number of authors [7-16]. However, all these applications use the Neumann expansion in a conventional way. Recently, Avila and co-workers [17-20] shed new light on the method, presenting two results for speeding up solution of uncertainty propagation problems using the Neumann expansion. In this paper, these results are reviewed and compared, and synergies between the two results are explored. Three derived, combined methods are also proposed.

2 MATHEMATICAL FORMULATION

2.1 Uncertainty propagation

In this paper, linear elliptic boundary value problemsare addressed; such operators appear in beam and plate bending problems, or in stationary head conduction problems. The stochastic uncertainty propagation problem is defined in a $\left(\Omega,\mathcal{F},P\right)$ probability space, where Ω is the sample space, $\mathcal{F}$ is a σ-algebra of events, and P is a probability measure. The linear stochastic uncertainty propagation problem solved herein is stated as:

$$\begin{cases} \text{Find } u\in L^2\left(\Omega,\mathcal{F},P;\left(H^{2m}\left(D\right)\cap H_0^m\left(D\right)\right)\right), \text{ such that} \\ \sum\limits_{|\alpha|,|\beta|\le m}\partial_\alpha\left(\kappa_{\alpha\beta}\partial_\beta\right)u\left(x,\omega\right)=f\left(x,\omega\right),\forall\left(x,\omega\right)\in D\times\left(\Omega,\mathcal{F},P\right),\text{a.e.}; \\ \text{subject to boundary conditions.} \end{cases} \tag{1}$$

where $f\left(\cdot,\cdot\right)$ is a source term. The random character of the solution is given by the set of coefficients $\kappa_{\alpha\beta}$. In solid mechanics problems, these coefficients can be associated to stiffness or thermal conductivity. In order to warrant existence and uniqueness of the solution to Eq. (1), somehypotheses are required on its coefficients, as detailed in [2-6].

In this paper, the Galerkin method is used to obtain approximated numerical solutions based on the Abstract Variational Problem (AVP) derived from Eq. (1). Details can be found in [17-20]. With the Galerkin formulation, and for the k^{th} realization of system parameters, a linear system of algebraic equations is obtained:

$$\begin{cases} \text{For fixed}\left\{\xi_{\alpha\beta}\left(\omega_k\right)\right\}\text{find } \mathcal{U}\left(\xi_{\alpha\beta}\left(\omega_k\right)\right)\in\mathbb{R}^m \text{ such that :} \\ \mathcal{K}\left(\xi_{\alpha\beta}\left(\omega_k\right)\right)\mathcal{U}\left(\xi_{\alpha\beta}\left(\omega_k\right)\right)=\mathcal{F}; \end{cases} \tag{2}$$

where $\mathcal{K}\left(\xi_{\alpha\beta}\left(\omega_k\right)\right) \in \mathbb{M}_m\left(\mathbb{R}\right)$ is a stiffness matrix, $\mathcal{F} \in \mathbb{R}^m$ is the source term or load vector, and $\mathcal{U}\left(\xi_{\alpha\beta}\left(\omega_k\right)\right)$ is a vector whose entries are coefficients of a linear combination.In Eq. (2), $\xi_{\alpha\beta}\left(\omega_k\right)$ are random vectors which represent the uncertainty in $\kappa_{\alpha\beta}$. Formally, solution to the linear system of equations in Eq. (2) is given by:

$$\mathcal{U}\left(\xi_{\alpha\beta}\right) = \left(\mathcal{K}\left(\xi_{\alpha\beta}\right)\right)^{-1} \mathcal{F} = \mathcal{H}\left(\xi_{\alpha\beta}\right)\mathcal{F}, \tag{3}$$

where matrix $\mathcal{H}\left(\xi_{\alpha\beta}\right) = \left[h_{ij}\left(\xi_{\alpha\beta}\right)\right]_{m\times m}$ is the inverse of stiffness matrix $\mathcal{K}\left(\xi_{\alpha\beta}\right)$. The approximated Galerkin solution becomes:

$$u_m\left(x,\xi_{\alpha\beta}\right) = \sum_{i=1}^{m}\sum_{j=1}^{m}\left(h_{ij}\left(\xi_{\alpha\beta}\right) f_j\right)\varphi_i\left(x\right) = \left(\mathcal{F}\,\mathcal{H}\left(\xi_{\alpha\beta}\right)\right)\cdot\mathbf{\Phi}\left(x\right), \tag{4}$$

where $\mathbf{\Phi}: D \rightarrow \mathbb{R}^m$ is a vector, whose entries are the Galerkin interpolating functions.

Based on Monte Carlo simulation, estimates for expected value and variance of random system response are obtained from the ensemble of system realizations, $\left\{u_m\left(x,\xi_{\alpha\beta}\right)\right\}_{k=1}^{N}$. For each realization of system response (Eq. 4), the inverse of the stiffness matrix, $\mathcal{H}\left(\cdot\right)$, needs to be computed. Iterative methods, such as Jacobi, Gauss-Seidell, Conjugated Gradients, among others, can be used [21]. However, the direct Monte Carlo solution can become prohibitive, due to the large number of system realizations which need to be computed. Under certain conditions, one alternative to reduce the computational burden is use of the Neumann series.

2.2 The Neumann series

The version of the Neumann series to be employed in this article refers to a linear operator, defined in a space of finite dimension. This operator is the stiffness matrix $\mathcal{K}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ which, for the k^{th} sample of coefficients, admits the following decomposition:

$$\mathcal{K}\left(\xi_{\alpha\beta}\right) = \mathcal{K}_0 - \Delta\mathcal{K}\left(\xi_{\alpha\beta}\right) \quad \Rightarrow \quad \begin{cases} \mathcal{K}\left(\xi_{\alpha\beta}\right) = \mathcal{K}_0\left(\mathcal{I} - \mathcal{P}\left(\xi_{\alpha\beta}\right)\right), \\ \mathcal{P}\left(\xi_{\alpha\beta}\right) = \left(\mathcal{K}_0\right)^{-1}\Delta\mathcal{K}\left(\xi_{\alpha\beta}\right); \end{cases} \tag{5}$$

where $\mathcal{I}$ is the identity matrix, $\mathcal{P}\left(\xi_{\alpha\beta}\right) \in \mathbb{M}_m\left(\mathbb{R}\right)$ and $\mathcal{P}\left(\xi_{\alpha\beta}\right) = \left[p_{ij}\left(\xi_{\alpha\beta}\right)\right]_{m\times m}$. The entries of matrix $\mathcal{K}_0$ are evaluated from expected values of the coefficients, $\mu_{\kappa_{\alpha\beta}}$. Replacing Eq. (5) in Eq. (4), a formal solution $\mathcal{U} = \mathcal{U}\left(\xi_{\alpha\beta}\right)$ is obtained as:

$$\mathcal{U}\left(\xi_{\alpha\beta}\right) = \left(\mathcal{K}\left(\xi_{\alpha\beta}\right)\right)^{-1}\mathcal{F} = \left(\mathcal{I} - \mathcal{P}\left(\xi_{\alpha\beta}\right)\right)^{-1}\mathcal{U}_0, \tag{6}$$

With $\mathcal{U}_0 = \left(\mathcal{K}_0\right)^{-1}\mathcal{F}$ and $\mathcal{U}_0 = \left[u_1^0,\ldots,u_m^0\right]^t$. From the Neumann series theorem [22], if $0 < \left\|\mathcal{P}\left(\xi_{\alpha\beta}\right)\right\| < 1$, then $\exists\left(\mathcal{I} - \mathcal{P}\left(\xi_{\alpha\beta}\right)\right)^{-1} \in \mathbb{M}_m\left(\mathbb{R}\right)$ and:

$$\left(\mathcal{I} - \mathcal{P}\left(\xi_{\alpha\beta}\right)\right)^{-1} = \sum_{q=0}^{\infty}\left(\mathcal{P}\left(\xi_{\alpha\beta}\right)\right)^q. \tag{7}$$

Truncating the Neumann seriesin the n^{th} term, and after some manipulations, one obtains the following approximate solution:

$$u_{mn}\left(x,\boldsymbol{\xi}_{\alpha\beta}\right)=\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{q=0}^{n}\left(p_{ij}^{(q)}\left(\boldsymbol{\xi}_{\alpha\beta}\right)u_j^0\right)\varphi_i\left(x\right)=\left(\mathcal{U}_0\right)^T\sum_{q=0}^{n}\left(\boldsymbol{\mathcal{P}}\left(\boldsymbol{\xi}_{\alpha\beta}\right)\right)^q\boldsymbol{\Phi}\left(x\right). \tag{8}$$

3 RECENT DEVELLOPMENTS

3.1 Bounds on samples of system response

Based on properties of the Neumann expansion and classical results of algebra, a methodology was proposed in [17,18] to derive lower and upper bounds for samples of the response process. The main ideas were: to avoid direct computation of the linear system of equations (Eq. 4) and to obtain an approximated solution, $u_{mn}=u_{mn}\left(x,\boldsymbol{\xi}_{\alpha\beta}\right)$, using a truly small ($n$=0 or n=1) number of terms in the Neumann expansion.

If the set of parameter samples, $\left\{\kappa_{\alpha\beta}\left(x,\boldsymbol{\xi}_{\alpha\beta}\left(\omega_k\right)\right)\right\}_{k=1}^{N}$, are such that $0<\left\|\boldsymbol{\mathcal{P}}\left(\boldsymbol{\xi}_{\alpha\beta}\left(\omega_k\right)\right)\right\|<1,\forall k\in\left\{1,...,N\right\}$; then the numerical approximations to the response fields can be represented in terms of the Neumann series using matrix $\left(\boldsymbol{\mathcal{I}}+\boldsymbol{\mathcal{P}}\left(\boldsymbol{\xi}_{\alpha\beta}\right)\right)^{-1}$. For the k^{th} response realization, and for a fixed $x\in\overline{D}$, the distance between response realizations obtained by direct Monte Carlo, Eq. (4), and by the Neumann series, Eq. (7), becomes:

$$\left|\left(u_m-u_{mn}\right)\left(x,\boldsymbol{\xi}_{\alpha\beta}\right)\right|\leq\sum_{q=n+1}^{\infty}\left\|\boldsymbol{\mathcal{P}}\left(\boldsymbol{\xi}_{\alpha\beta}\right)\right\|^q\left\|\mathcal{U}_0\right\|\left\|\boldsymbol{\Phi}\left(x\right)\right\|. \tag{9}$$

Based on properties of the Neumann series [17-20],this distance can be written as:

$$\left|\left(u_m-u_{mn}\right)\left(x,\boldsymbol{\xi}_{\alpha\beta}\right)\right|\leq\lambda\left(n,\boldsymbol{\xi}_{\alpha\beta}\right)\left\|\mathcal{U}_0\right\|\left\|\boldsymbol{\Phi}\left(x\right)\right\|;\text{ where }\lambda\left(n,\boldsymbol{\xi}_{\alpha\beta}\right)=\frac{\left\|\boldsymbol{\mathcal{P}}\left(\boldsymbol{\xi}_{\alpha\beta}\right)\right\|^{n+1}}{1-\left\|\boldsymbol{\mathcal{P}}\left(\boldsymbol{\xi}_{\alpha\beta}\right)\right\|}. \tag{10}$$

Eq. (10) can be rewritten to yield lower $\alpha\left(\cdot,\cdot\right)$ and upper $\beta\left(\cdot,\cdot\right)$ bounds, for the k^{th} sample of the response process:

$$\alpha\left(x,\boldsymbol{\xi}_{\alpha\beta}\right)\leq u_m\left(x,\boldsymbol{\xi}_{\alpha\beta}\right)\leq\beta\left(x,\boldsymbol{\xi}_{\alpha\beta}\right),\forall\left(x,\boldsymbol{\xi}_{\alpha\beta}\right)\in\overline{D}\times\left(\Omega,\mathcal{F},P\right); \tag{12}$$

with:

$$\begin{cases}\alpha\left(x,\boldsymbol{\xi}_{\alpha\beta}\right)=u_{mn}\left(x,\boldsymbol{\xi}_{\alpha\beta}\right)+\lambda\left(n,\boldsymbol{\xi}_{\alpha\beta}\right)\left\|\mathcal{U}_0\right\|\left\|\boldsymbol{\Phi}\left(x\right)\right\|,\\ \beta\left(x,\boldsymbol{\xi}_{\alpha\beta}\right)=u_{mn}\left(x,\boldsymbol{\xi}_{\alpha\beta}\right)-\lambda\left(n,\boldsymbol{\xi}_{\alpha\beta}\right)\left\|\mathcal{U}_0\right\|\left\|\boldsymbol{\Phi}\left(x\right)\right\|,\forall\left(x,\boldsymbol{\xi}_{\alpha\beta}\right)\in\overline{D}\times\left(\Omega,\mathcal{F},P\right).\end{cases} \tag{13}$$

With the functions in Eq. (13), and from the ensemble of realizations of lower and upper bounds of system realizations, $\left\{\alpha\left(x,\boldsymbol{\xi}_{\alpha\beta}\left(\omega_k\right)\right),\beta\left(x,\boldsymbol{\xi}_{\alpha\beta}\left(\omega_k\right)\right)\right\}_{k=1}^{N}$, it is possible to obtain lower and upper bounds for expected value and autocorrelation of the response process.

The coefficient $\lambda\left(\cdot,\cdot\right)$ in aboveequationsdepends on the number of terms ("n") used in the Neumann series for matrix $\left(\boldsymbol{\mathcal{I}}+\boldsymbol{\mathcal{P}}\left(\boldsymbol{\xi}_{\alpha\beta}\right)\right)^{-1}$, and on the computation for the norm $\left\|\boldsymbol{\mathcal{P}}\right\|$. Using the Euclidian norm, the computational cost can become quite large, turning the procedure useless. In order to circumvent this difficulty, the authors [17,18] resorted to the

equivalence between matrix norms [22]. Based on this equivalence, five alternatives were proposed in [17,18] to evaluate $\|\mathcal{P}\|$ and the coefficient "λ": norm one, theEuclidian norm, the infinity norm, theFrobenius norm and the maximum norm. Results obtained for beam bending in [17], and for axial thermo-elasticity in [17, 18], for random fields $\xi_{\alpha\beta}$ with coefficient of variation of 10%, and using one or two terms in the Neumann expansion (n=0 or n=1), confirms that using the Euclidian norm leads to the most accurate, yet computationally expensive results. A good compromise between accuracy and efficiency was obtained using the Frobenius norm, with errors in expected values between -2% and 3%, and errors in auto-correlation between -4% and 6%.

3.2 The lambda (λ) convergence factor

Based on the solution of a distance minimization problem, very accurate and efficient results were obtained in [19, 20] using a low-order (n=1) Monte Carlo-Neumann solution. The distance (error) to be minimized was called λ; hence the methodology was called the Monte Carlo-Neumannwith λ convergence, or simply MC-N λ method. A one-dimensional version of the methodology was presented in [19]. The most accurate, two-dimensional version [20] is reproduced here.

Consider the k^{th} sample of coefficients $\kappa_{\alpha\beta}\left(x,\xi_{\alpha\beta}\left(\omega_k\right)\right)$ such that $0<\left\|\mathcal{P}\left(\xi_{\alpha\beta}\left(\omega_k\right)\right)\right\|<1$. From Eq. (7), the n^{th} order representation, via Neumann series, of the inverse is given by,

$$\left(\mathcal{I}-\mathcal{P}\left(\xi_{\alpha\beta}\right)\right)^{-1}_{(n)}=\sum_{q=0}^{n}\left(\mathcal{P}\left(\xi_{\alpha\beta}\right)\right)^{q}. \tag{14}$$

From the definitions in Eqs. (6) and (14), the inverse of the stiffness matrix is given by:

$$\left(\mathcal{K}\right)^{-1}_{(n)}=\left(\mathcal{I}-\mathcal{P}\left(\xi_{\alpha\beta}\right)\right)^{-1}_{(n)}\left(\mathcal{K}_0\right)^{-1}=\sum_{q=0}^{n}\left(\mathcal{P}\left(\xi_{\alpha\beta}\right)\right)^{q}\left(\mathcal{K}_0\right)^{-1}. \tag{15}$$

It is evidentthat $\left(\mathcal{K}\left(\xi_{\alpha\beta}\right)\right)^{-1}\mathcal{K}\left(\xi_{\alpha\beta}\right)=\mathcal{I}$; however, this identity does not hold exactly when an approximation of the inverse, such as Eq. (15), is used. Hence, one can write:

$$\left(\mathcal{K}\left(\xi_{\alpha\beta}\right)\right)^{-1}_{(n)}\mathcal{K}\left(\xi_{\alpha\beta}\right)=\left(\mathcal{I}-\mathcal{P}\left(\xi_{\alpha\beta}\right)\right)^{-1}_{(n)}\left(\mathcal{I}-\mathcal{P}\left(\xi_{\alpha\beta}\right)\right)=\mathcal{I}-\mathcal{E}_{(n)}\left(\xi_{\alpha\beta}\right). \tag{16}$$

In Eq. (16), matrix $\mathcal{E}_{(n)}\in\mathbb{M}_m\left(\mathbb{R}\right)$ reveals how much the inverse approximation $\left(\mathcal{K}\left(\xi_{\alpha\beta}\right)\right)^{-1}_{(n)}$, when multiplied by the true inverse, departs from the identity matrix. Since use of the Neumann series is motivated by reduction of computation cost, it was proposed in [19, 20] to formulate the problem for the first order Neumann approximation, with n=1. Nevertheless, higher order approximations could be pursued as well. For the first order approximation, and after some algebra, the following distance minimization problem can be stated [20]:

$$\begin{cases}\text{Find } \left(\lambda_1^*,\lambda_2^*\right)\in\mathbb{R}^2 \text{ such that}\\ \left(\lambda_1^*,\lambda_2^*\right)=\underset{(\lambda_1,\lambda_2)\in\mathbb{R}^2}{\arg\min}\left\{\left(\tfrac{1}{2}\right)\left\|\left(\lambda_1\left(\mathcal{I}-\mathcal{P}\left(\xi_{\alpha\beta}\right)\right)+\lambda_2\mathcal{P}\left(\xi_{\alpha\beta}\right)\left(\mathcal{I}-\mathcal{P}\left(\xi_{\alpha\beta}\right)\right)-\mathcal{I}\right)\mathcal{U}_0\right\|^2\right\}.\end{cases} \tag{19}$$

where $\|\cdot\|$ is the Euclidian norm in $\mathbb{R}^n$. The objective function in Eq. (19) is non-negative and convex; hence the global optimum is obtained from stationarity conditions [23]:

$$\left(\nabla_{\lambda} f\right)\left(\lambda_1^*,\lambda_2^*\right)=\mathbf{0}\quad\Rightarrow\quad\lambda_1^*=\frac{\phi-\varphi\lambda_2^*}{\alpha};\quad\lambda_2^*=\frac{\alpha\vartheta-\varphi\phi}{\alpha\gamma-\varphi^2}; \tag{20}$$

where

$$\begin{cases} \alpha(\xi_{\alpha\beta}) = \left(\left(\mathcal{I} - \mathcal{P}(\xi_{\alpha\beta})\right)\mathcal{U}_0\right)^T \left(\mathcal{I} - \mathcal{P}(\xi_{\alpha\beta})\right)\mathcal{U}_0; \\ \varphi(\xi_{\alpha\beta}) = \left(\left(\mathcal{I} - \mathcal{P}(\xi_{\alpha\beta})\right)\mathcal{U}_0\right)^T \mathcal{P}(\xi_{\alpha\beta})\left(\mathcal{I} - \mathcal{P}(\xi_{\alpha\beta})\right)\mathcal{U}_0; \\ \phi(\xi_{\alpha\beta}) = \mathcal{U}_0^T \left(\mathcal{I} - \mathcal{P}(\xi_{\alpha\beta})\right)\mathcal{U}_0; \\ \gamma(\xi_{\alpha\beta}) = \left(\mathcal{P}(\xi_{\alpha\beta})\left(\mathcal{I} - \mathcal{P}(\xi_{\alpha\beta})\right)\mathcal{U}_0\right)^T \left(\mathcal{P}(\xi_{\alpha\beta})\left(\mathcal{I} - \mathcal{P}(\xi_{\alpha\beta})\right)\mathcal{U}_0\right); \\ \vartheta(\xi_{\alpha\beta}) = \mathcal{U}_0^T \left(\mathcal{P}(\xi_{\alpha\beta})\left(\mathcal{I} - \mathcal{P}(\xi_{\alpha\beta})\right)\mathcal{U}_0\right). \end{cases} \tag{21}$$

Hence, by the proposed MC-N λ method, with n=1,the k^{th} realization of system response is:

$$\mathcal{U}_{(1)}(\lambda^*, \xi_{\alpha\beta}) = \left(\lambda_1^* \mathcal{I} + \lambda_2^* \mathcal{P}(\xi_{\alpha\beta})\right)\mathcal{U}_0. \tag{22}$$

Replacing Eq. (22) in Eq. (8), one obtains:

$$\begin{aligned} u_{m1}(x, \lambda^*, \xi_{\alpha\beta}) &= \lambda_1^* \sum_{i=1}^{m}\sum_{j=1}^{m} u_j^0 \varphi_i(x) + \lambda_2^* \sum_{i=1}^{m}\sum_{j=1}^{m} p_{ij}(\xi_{\alpha\beta}) u_j^0 \varphi_i(x) \\ &= (\Phi(x))^t \mathcal{U}_{(1)}(\lambda^*, \xi_{\alpha\beta}) = (\Phi(x))^t \left(\lambda_1^* \mathcal{I} + \lambda_2^* \mathcal{P}(\xi_{\alpha\beta})\right)\mathcal{U}_0, \end{aligned} \tag{23}$$

where $u_{m1}(\cdot, \lambda^*, \cdot)$ is the numerical approximation of the solution, based on the first order λ convergence parameter.

The Neumann-λmethod proposed in [19, 20] was shown to lead to almost exact solutions, with very small computational costs. In [19], the one-dimensional version of the method was applied to axial reaction-diffusion problems. Considering a random membrane stiffness, with coefficient of variation of up to 30%, the error in expected value was of 0.7%, and the error in variance was of up to 4%. These errors were shown to be smaller than the errors for a conventional, Monte Carlo - Neumann solution with n=5, but much cheaper to compute.

In ref. [20], the two-dimensional version of the method, presented in this section, was applied to stochasticEulerBernoullibeambendingproblems. The coefficient of variation of beam bending stiffness was up to 30%. The errors in expected value and variance were shown to be smaller than 0.1% and 1%, respectively; these errors are much smaller than errors of a conventional MC-Neumann solution with n=5, yet faster to compute!

4 NEW DEVELOPMENTS AND RESULTS

Kist [24] investigated possible synergies between the results presented in Section 3. The author developed one variant of the bounding method, and two new methods combining the Neumann bounds with the λ convergence parameter. These three methods, plus the original two, were applied to problems of stochastic Kirchhoff plate bending.

The first result employs the Cauchy-Schwartz inequality to the difference between the Galerkin solution (Eq. 4) and its Neumann series approximation (Eq. 8), as stated in Eq. (9). Following a development similar to Section 3.1, the author arrives at improved bounds for the realizations of system responses.

The second result in [24] assumes that the first order Neumann series can be approximated by the one-dimensional approximation of ref. [19]:

$\left(\mathcal{I}-\mathcal{P}\left(\xi_{\alpha\beta}\right)\right)^{-1}_{(1)}=\mathcal{I}+\alpha^{*}\mathcal{P}\left(\xi_{\alpha\beta}\right)$, where α^{*} is a parameter to be determined. This solution leads to different bounds for system response, following section 3.1, and is called Mixed 1D in the comparisons below.

The third result by [24] assumes that the first order Neumann series can be approximated by: $\left(\mathcal{I}-\mathcal{P}\left(\xi_{\alpha\beta}\right)\right)^{-1}_{(1)}=\lambda_1^{*}\mathcal{I}+\lambda_2^{*}\mathcal{P}\left(\xi_{\alpha\beta}\right)$, following [20], and uses this result to construct bounds for system response, following section 3.1. This solution is called Mixed2D in the sequence.

The three methods developed in [24], plus the two original methods in [17-20] were applied in [24] to the solution of stochastic Kirchhoff plate bending problems, withWinkler and Pasternak-type foundations. These results are summarized in Table 1, which gives further insight into the methods. The results refer to problem variants with uncertain plate and/or foundation stiffness, and with different coefficients of variation ($\delta=\sigma/\mu$). All error measures reported herein are maximum errors obtained in the problem domain, computed in comparison to direct Monte Carlo simulation. For all bounding solutions, results are reported for the Euclidian norm. Computation times for bound solutions were similar between Euclidian and Frobenius norms, and approximately one-third of the computation time for direct MC simulation. Also, results obtained for the three methods proposed in [24] were similar when computed using Euclidian or Frobenius norms.

Table 1: Comparison of results for five different solutions of plate bending problems.

Problem	Solution method	Error in Mean (%)	Error in Variance (%)
Random plate stiffness, $\delta_\kappa=0.1$, Winkler foundation	**MC-N-λ**	**0.19**	**0.04**
	Bounds	± 2.0	± 16.0
	Improved bounds	± 2.0	± 4.0
	Mixed 1D	± 0.5	± 0.9
	Mixed 2D	± 0.2	± 0.5
Random foundation stiffness, $\delta_{\kappa w}=0.5$, Winkler foundation	**MC-N-λ**	**0.9 10^{-6}**	**1.8 10^{-6}**
	Bounds	± 330 10^{-6}	± 700 10^{-6}
	Improved bounds	± 210 10^{-6}	± 400 10^{-6}
	Mixed 1D	± 12.0 10^{-6}	± 24.0 10^{-6}
	Mixed 2D	± 4.0 10^{-6}	± 8.0 10^{-6}
Random plate stiffness, $\delta_\kappa=0.1$, Pasternak foundation	**MC-N-λ**	**0.05**	**0.1**
	Mixed 2D	± 0.25	± 0.5
Pasternak foundation, random plate and foundation stiffness, $\delta_\alpha=\delta_{\kappa p}=\delta_{\kappa w}=0.1$,	**MC-N-λ**	**0.04**	**0.2**
	Bounds	± 9.0	± 18.0
	Improved bounds	± 2.0	± 4.0
	Mixed 1D	± 0.5	± 1.0
	Mixed 2D	± 0.25	± 0.5

First, it is observed that the Monte Carlo – Neumann λ method yields very accurate results for all plate-bending problems addresses in [24], also for a problem with very large Winkler foundation stiffness ($\delta_{\kappa w}=0.5$). The original bounds proposed in [17,18] lead to large deviations from reference results, with errors of up to ±9% in the mean, and up to ±18%

in variances. Also, for the large foundation stiffness problem ($\delta_{kw} = 0.5$), the errors obtained via bounds are significantly larger than for the other methods. The improved bounds proposed in [24] provide better results, in comparison to the original, but the improvements are insignificant w.r.t. to quality of other solutions.

The mixed formulations provided good results but, again, the accuracy of such results is not as good as the accuracy of the original, two-dimensional version of the Monte Carlo – Neumann λ method [19,20]. The 2D mixed formulation provided more accurate solutions than its one-dimensional counterpart, as expected.

5 CONCLUSION

This paper reviewed two recent theoretical results, regarding application of the Neumann series in solution of uncertainty propagation problems in mechanics, and explored synergies between these solutions. The first result, based on unexplored properties of the Neumann series, allows establishing lower and upper bounds for the realizations of the response process.The second result derives from a minimization of the error obtained by multiplying the true stiffness matrix by a first-order Neumann approximation of its inverse. The deviation of this internal product from the identity matrix yields aλ factor, which can be solved analytically to yield very accurate and low-cost solutions. Three derived results were also presented: improved bounds for system responses, and two mixed formulations, were one-dimensional and two-dimensional λ factors are used to compute response bounds. The five methods were employed in solution of stochastic plate bending problem, with Winkler and Pastenak types of foundation. Results show that the improved bounds are better than the original, but still present large errors in mean and standard deviations of the response,for some problems. The mixed formulations provided interesting and accurate bounding results, in comparison to the original and improved bounds. However, these bounding solutions are not as accurate or efficient as the original 2D Neumann-λ method [19,20], which also yielded very accurate results for all plate bending problems addressed herein. These new results confirm that the Neumann-λ is a powerful method for the solution of uncertainty propagation problems in mechanics. Synergies between this and other methods of uncertainty propagation, such as perturbation, stochastic FEM, etc., are yet to be explored.

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