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EXACT STOCHASTIC ANALYSIS OFCOUPLED BENDING-TORSION BEAMS WITH IN-SPAN SUPPORTS AND MASSES

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Abstract. *The stochastic response of acoupled bending-torsion beam, carrying an arbitrary number of supports/masses, is investigated. Usingthe theory of generalized functions in conjunction with the elementary coupled bending-torsion beam theory, exact analytical solutions under stationary inputs are obtained based on frequency response functions derived by two different closed-form expressions. The analytical solutions are obtained for all response variables, considering any number of supports/masses along the beam and arbitrary spatial load distributions. A pertinentnumericalexampleisreported.*

Sommario. *Viene indagata la risposta stocastica di una trave con un numero arbitrario di supporti elastici e masse appese,che presenta un comportamento flessionale e torsionale accoppiato. Utilizzando la teoria delle funzioni generalizzate e la teoria elementare di accoppiamento flessionale-torsionale, soluzioni analitiche esatte sono ottenute, per eccitazioni stazionarie, attraverso funzioni di risposta in frequenza derivate attraverso due diverse espressioni in forma chiusa. Le soluzioni analitiche sono ottenute per tutte le variabili della risposta, considerando un numero arbitrario di supporti e masse lungo la trave e una qualsiasi distribuzione spaziale del carico. Si riportainfine unesempionumerico.*

1 INTRODUCTION

The response of beams to random loads has been widely investigated in literature, and some examples can be found in ref. [1-2]. Most works concernedbending vibration analysis based on Euler-Bernoulli or Timoshenko beam theory, under the fundamental assumption thatthe beam cross-section is doubly-symmetric, so that the shear center (SC) coincides with the gravity center (GC).

However, beams with mono-symmetric cross sections are frequently employed in different engineering applications such as wings, turbine blades and propellers. Because, in this case, the SC does coincide with the GC, the dynamics of these beams exhibits coupled bending-torsion phenomena. Indeed, bending-induced inertial forces, which arise along the mass axis (i.e. the locus of the gravity centers of the beam cross sections), are eccentric with respect to the elastic axis (i.e. the locus of the shear centers of the beam cross sections) and, as a result, bending is inherently coupled with twisting. The coupling effects have generally been described by the so-called elementary coupled bending-torsion beam theory [3-4], i.e. neglecting warping effects and not considering rotatory inertia and shear deformation of the beam. These effects were also considered in other studies [5-6].

As for forced vibration analysis, using the normal mode method Eslimy et al. [7-8] obtained the response of a coupled bending-torsion beam when subjected to deterministic as well asstationary, Gaussian loads. Specifically, they applied the method to a cantilever aircraft wing for which there is a substantial coupling between bending and torsional vibrations.

It must be noticed, however, that all works in ref.[1] through ref.[8] generally addressed uniform beams, with no attachments or in-span supports. The latter are of great interest for engineering applications but have rarely been considered in studies on coupled bending-torsion phenomena.

This paper proposes an exact method for beams with mono-symmetric cross section, carrying an arbitrary number of elastic supports and attached masses, subjected to stationary loads. The elementary bending-torsion theory is considered as in ref.[3-4], in conjunction with the theory of generalized functions to handle the discontinuities of the response variables at the application points of supports and masses. Exact analytical solutions for the response are built based on frequency response functions of the beam obtained, in this paper, by two different closed-form expressions. The key step to build the frequency response functions is a novel closed-form analytical expression derived, in this paper, forthe response of the beam without response supports/masses, subjected to arbitrarily-placed harmonic unit force and unit twisting moment. A numerical example will show the response power spectral density functionof a beam with angular section, carrying translational and torsional-rotational elastic supports, under a transverse stationary load.

2 PROBLEM STATEMENT

Figure 1 shows the case under study, i.e. a uniform straight beam of length*L*, referred to a right handed coordinate system, carrying an arbitrary number of elastic supports and attached masses and subjected to a transverse distributed load. Beam cross section is assumed to be mono-symmetric, being *x* its symmetric axis. The loci of the shear centers and mass centers of the beam cross section are respectively the elastic axis and the mass axis; the first coincides with the *y*-axis, while the latter is at distance x_a from *y*-axis. The bending deflection in the *z*-direction, the bending rotation about the *x*-axis and the torsional rotation about the *y*-axis of the shear centers are denoted respectively by $h(y,t), \theta(y,t)$ and $\psi(y,t)$.

The transverse distributed load, acting parallel to $O-z$, is applied at distance x_c from y-axis; it can be separated into a transverse load applied along the elastic axis, and a distributed torque about Oy, as shown in Figure 1. The elastic support and attached masses are applied at y_j, with $0 < y_1 < ... < y_j < L$ and stiffness parameters of the *j*-th support and properties of the *j*-th mass are denoted as follows:

- k_{H_j} for translational support, k_{Ψ_j} for torsional-rotational support and k_{θ_j} for bending rotational support.
- M_j is the mass, I_{xx_j} and I_{yy_j} the components of the mass inertia tensor about axes *x*

and y in Figure 1.

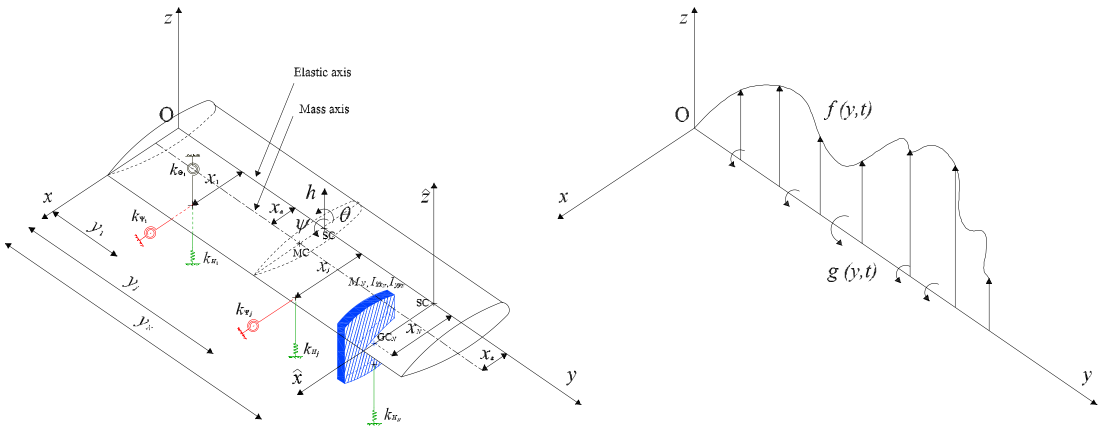

Figure 1: Beam with a mono-symmetric cross-section carrying an arbitrary number of elastic supports and attached masses and subjected to a transverse distributed load and distributed torque

2.1 Equations of motion

Pursuing the primary purpose of obtaining the frequency response functions in order to determine the stochastic response, assume that the beam, carrying supports/masses as shown in Figure 1, is loaded by a harmonic distributed load $f(y)e^{i\omega t}$, and a harmonic distributed torque $g(y)e^{i\omega t} = f(y)x_c e^{i\omega t}$, on the interval (a,b), with $0 \le a \le b \le L$, where ω is the frequency. Let be $h(y,\omega,t) = H(y,\omega)e^{i\omega t}, \psi(y,\omega,t) = \Psi(y,\omega)e^{i\omega t}$, the steady state response variables. By using the elementary coupled bending-torsion theory and making use of the generalized functions, the following steady state coupled equations of motion are derived:

$$EI\frac{\overline{d}^4 H}{dy^4} - (m\omega^2 - c_1 i\omega)H + (mx_a\omega^2 - c_1 x_a i\omega)\Psi - \sum_{j=1}^{N} P_j\delta(y-y_j) + \sum_{j=1}^{N} Mf_j\delta^{(1)}(y-y_j) + f(y) = 0 \qquad (1)$$

$$GJ\frac{\overline{d}^2 \Psi}{dy^2} - (m\omega^2 x_a - c_1 i\omega x_a)H + (I_\alpha\omega^2 - c_2 i\omega)\Psi - \sum_{j=1}^{N} P_j x_j\delta(y-y_j) + \sum_{j=1}^{N} Mt_j\delta^{(1)}(y-y_j) + g(y) = 0 \qquad (2)$$

where EI and GJ are respectively bending and torsional rigidities, m is the mass per unit length, I_α is the polar moment of inertia per unit length about the elastic axis, while c_1 and c_2 are viscous damping coefficient per unit length respectively in bending and in torsion, to be assigned so that damping is proportional [8]; in Eqs.(1)-(2) $\delta^{(k)}(y-y_j)$ is the k-th formal derivative of Dirac's delta $\delta(y-y_j)$, while P_j, Mt_j and Mf_j are concentrated force, twisting moment and bending moment associated with supports and attached masses at y_j, and they are given as:

$$P_j = -\kappa_{Pj}(\omega)\left[H(y_j) - x_j\Psi(y_j)\right] \quad ; \quad Mt_j = -\kappa_{Tj}(\omega)\Psi(y_j) \quad ; \quad Mf_j = -\kappa_{Mj}(\omega)\Theta(y_j) \qquad (3)$$

being $H(y_j), \Psi(y_j)$ and $\Theta(y_j)$ the deflection, torsional and bending rotation at $y = y_j$, and $\kappa_{Pj}(\omega), \kappa_{Tj}(\omega), \kappa_{Mj}(\omega)$ frequency dependent terms given as:

$$\kappa_{Pj}(\omega) = \kappa_{Hj} - M_j\omega^2 \quad ; \quad \kappa_{Tj} = k_{\Psi j} - (I_{yy_j} - M_j x_j^2)\omega^2 \quad ; \quad \kappa_{M_j}(\omega) = k_{\Theta_j} - I_{xx_j}\omega^2 \qquad (4)$$

Eqs.(1)-(2)can be combined, by eliminating either Hor Ψ, to obtain two uncoupled 6-th order differential equations for deflection and torsional rotation:

$$\alpha\frac{\overline{d}^{6}H}{dy^{6}}+\beta\frac{\overline{d}^{4}H}{dy^{4}}-\gamma\frac{\overline{d}^{2}H}{dy^{2}}+\eta H-(\frac{I_{a}\omega^{2}}{mx_{a}\omega^{2}}-x_{j})f(y)-\frac{GJ}{mx_{a}\omega^{2}}f^{(2)}(y)-g(y)+R_{H_{ext}}(y)=0 \tag{5}$$

$$\alpha\frac{\overline{d}^{6}\Psi}{dy^{6}}+\beta\frac{\overline{d}^{4}\Psi}{dy^{4}}-\gamma\frac{\overline{d}^{2}\Psi}{dy^{2}}+\eta\Psi+\frac{EIx_{j}}{mx_{a}\omega^{2}}f^{(4)}(y)-\frac{x_{j}-x_{a}}{x_{a}}f(y)+\frac{EI}{mx_{a}\omega^{2}}g^{(4)}(y)-\frac{m\omega^{2}}{mx_{a}\omega^{2}}g(y)+R_{\Psi_{ext}}(y)=0 \tag{6}$$

where

$$\begin{aligned}R_{H_{ext}}(y)=&-\sum_{j=1}^{N}P_{j}\left[(\frac{I_{\alpha}\omega^{2}}{mx_{a}\omega^{2}}-x_{j})\delta(y-y_{j})+\frac{GJ}{mx_{a}\omega^{2}}\delta^{(2)}(y-y_{j})\right]-\sum_{j=1}^{N}Mt_{j}\delta(y-y_{j})+\\&+\sum_{j=1}^{N}Mf_{j}\left[\frac{I_{\alpha}\omega^{2}}{mx_{a}\omega^{2}}\delta^{(1)}(y-y_{j})+\frac{GJ}{mx_{a}\omega^{2}}\delta^{(3)}(y-y_{j})\right]\end{aligned} \tag{7}$$

$$\begin{aligned}R_{H_{ext}}(y)=&-\sum_{j=1}^{N}P_{j}\left[\frac{EIx_{j}}{mx_{a}\omega^{2}}\delta^{(4)}(y-y_{j})-\frac{x_{j}-x_{a}}{x_{a}}\delta(y-y_{j})\right]+\\&+\sum_{j=1}^{N}Mt_{j}\left[\frac{EI}{mx_{a}\omega^{2}}\delta^{(4)}(y-y_{j})-\frac{m\omega^{2}}{mx_{a}\omega^{2}}\delta(y-y_{j})\right]+\sum_{j=1}^{N}Mf_{j}\delta^{(1)}(y-y_{j})\end{aligned} \tag{8}$$

2.2 Solutions of the equations of motion

Consider first the following equation:

$$\alpha\frac{\overline{d}^{6}X}{dx^{6}}+\beta\frac{\overline{d}^{4}X}{dx^{4}}-\gamma\frac{\overline{d}^{2}X}{dx^{2}}+\eta X-\delta(y-y_{0})=0 \tag{9}$$

The solution *X*takes the form:

$$X(y,y_{0})=\sum_{j=1}^{6}\Omega_{j}c_{j}+J^{(*)}(y,y_{0}) \tag{10}$$

where

$$\begin{aligned}&\Omega_{1}=\cosh(\sqrt{r_{1}}X);\ \Omega_{2}=\sinh(\sqrt{r_{1}}X);\ \Omega_{3}=\cos(\sqrt{r_{2}}X);\\&\Omega_{4}=\sin(\sqrt{r_{2}}X);\ \Omega_{5}=\cos(\sqrt{r_{3}}X);\ \Omega_{6}=\sin(\sqrt{r_{1}}X);\end{aligned} \tag{11}$$

while the particular integral is obtained by *Mathematica*, after few manipulations in the following form:

$$\begin{aligned}J^{(*)}(y,y_{0})=D\Big[&\sinh(\sqrt{r_{1}}(y-y_{0}))\sqrt{r_{2}}\sqrt{r_{3}}(r_{3}-r_{2})-\sin(\sqrt{r_{2}}(y-y_{0}))\sqrt{r_{1}}\sqrt{r_{3}}(r_{1}+r_{3})+\\&+\sin(\sqrt{r_{3}}(y-y_{0}))\sqrt{r_{1}}\sqrt{r_{2}}(r_{1}+r_{2})\Big]\cdot U(y-y_{0})\end{aligned} \tag{12}$$

being $U(\cdot)$ the Unit-Step function, $d=\alpha^{3}/(4\eta\beta^{3}+4\gamma^{3}\alpha+27\alpha^{2}\eta^{2}-\gamma^{2}\beta^{2}-18\alpha\beta\gamma\eta)$ and $D=-d(r_{1}+r_{2})(r_{1}+r_{3})(r_{3}-r_{2})\big/\sqrt{r_{1}}\sqrt{r_{2}}\sqrt{r_{3}}$.

Starting from Eq.(9) closed-form solutions can readily be derived for Eqs.(5)-(6), as indeed the particular integrals associated with formal derivatives of the Dirac's delta, which are related to unit force and moments in Eqs.(5)-(6), can readily be obtained from Eq.(9) by successive derivation.

3 FREQUENCY RESPONSE FUNCTION OF BEAMS WITH SUPPORTS/MASSES VIA GENERALIZED FUNCTIONS

In this Section an exact method to build the frequency response function of the beam with supports/masses in Figure 1 will be shown.

Firstly considerthe beam in Figure 1 subjected only to the transverse distributed load $f(y)$ in the interval (a,b) with $0 \le a \le b \le L$. By applying the linear superposition principle, the vector $\mathbf{Y}^{(f)}(y) = \left[H^{(f)} \quad \Theta^{(f)} \quad M^{(f)} \quad S^{(f)} \quad \Psi^{(f)} \quad T^{(f)} \right]$ collecting all response variables of the beam with supports/masses can be written as:

$$\mathbf{Y}^{(f)}(y) = \mathbf{\Omega}(y)\mathbf{c}^{(f)} + \sum_{j=1}^{n} \mathbf{J}(y, y_j)\mathbf{\Lambda}_j^{(f)} + \mathbf{Y}_{(f)}(y) \tag{13}$$

In Eq.(13), vector $\mathbf{\Lambda}_j^{(f)} = \left[P_j \quad Mt_j \quad Mf_j \right]^T$ collects the unknown reaction force $P_j(\omega)$, twisting moment $Mt_j(\omega)$ and bending moment $Mf_j(\omega)$ at location y_j; $\mathbf{J}(y, y_j)$ is a 6×3 matrix collecting the particular integrals related to the response discontinuities at location y_j, built as explained in Section 2.2; $\mathbf{Y}_{(f)}(y)$ are the particular integrals related to load $f(y)$,

$$\mathbf{Y}_{(f)}(y) = \int_a^b \mathbf{J}^{(P)}(y,\xi) f(\xi) d\xi \tag{14}$$

where $\mathbf{J}^{(P)}(y,\xi)$ is the vector of the particular integrals related to a unit transverse load.

Through a recursive procedure, the unknowns $\mathbf{\Lambda}_j$ in Eq.(13)can be obtained as functions the vector of integration constants $\mathbf{c}^{(f)}$ only, to finally derive the following expression for $\mathbf{Y}^{(f)}(y)$

$$\mathbf{Y}^{(f)}(y) = \widetilde{\mathbf{Y}}(y)\mathbf{c}^{(f)} + \widetilde{\mathbf{Y}}_{(f)}(y) \tag{15}$$

where $\widetilde{\mathbf{Y}}(y)$ is a vector depending on the beam parameters only, and $\widetilde{\mathbf{Y}}_{(f)}(y)$ depends on the beam parameters and the applied load (expressions are omitted for brevity). The vector of integration constants $\mathbf{c}^{(f)}$ can be determined on setting the beam boundary conditions. Specifically, it will be obtained by inverting a 6×6 matrix in a closed form using Mathematica.

An alternative exact expression of the frequency response functions can be built by modal superposition. For this, exact natural frequenciescan be obtained from the eigenproblem

$$\mathbf{B}(\omega)\mathbf{c} = \mathbf{0} \tag{16}$$

where $\mathbf{B}$ is a 6×6 matrix built based on Eq.(13),where the load dependent term $\mathbf{Y}^{(f)}(y) = \mathbf{0}$ and $c_1 = c_2 = 0$ in Eqs.(1)-(2). Upon computing the natural frequenciesfrom Eq.(16), corresponding exact eigenfunctions are derived, in a closed analytical form, from Eq.(13)for $\mathbf{Y}^{(f)}(y)$ with, again, $\mathbf{Y}^{(f)}(y) = \mathbf{0}$ and $c_1 = c_2 = 0$ in Eqs.(1)-(2).Denoting H_n and Ψ_n as the eigenfunctions of deflection and torsional rotation, respectively,and by considering the orthogonality condition,derived as similarly in ref.[8], the following expression can be derived for the frequency response function for bending deflection and torsional rotation

$$H^{(f)}(y,\omega) = \sum_{n=1}^{\infty} q_n^{(f)}(\omega) H_n(y) \quad ; \quad \Psi^{(f)}(y,\omega) = \sum_{n=1}^{\infty} q_n^{(f)}(\omega) \Psi_n(y) \tag{17}$$

where $q_n^{(f)}(\omega)$ denotes the modal frequency response function of the n^{th} mode:

$$q_n^{(f)}(\omega)=\frac{\int_0^L f(y)\left[H_n(y)\right]dy}{\mu_n({\omega_n}^2-\omega^2+2i\xi_n\omega_n\omega)} \tag{18}$$

Eq.(18)provides an insight into the contributions of every mode to the frequency response of the beam, very useful for design purposes.

Frequency response function for the other response variables can be obtained from Eq.(17) by considering Saint-Venant and Euler-Bernoulli beam theories.

Similarly the frequency response functions $\mathbf{Y}^{(g)}(y)$ due tothe torque $g(y)=f(y)x_c$, can be obtained replacing in Eq.(13) $\mathbf{Y}_{(f)}(y)$ withthe particular integral $\mathbf{Y}_{(g)}(y)$,given as

$$\mathbf{Y}_{(g)}(y)=\int_a^b \mathbf{J}^{(Mt)}(y,\xi)g(\xi)d\xi \tag{19}$$

and replacing in Eq.(17) $q_n^{(f)}(\omega)$ with $q_n^{(g)}(\omega)$, whose expression is:

$$q_n^{(g)}(\omega)=\frac{\int_0^L g(y)\left[\Psi_n(y)\right]dy}{\mu_n({\omega_n}^2-\omega^2+2i\xi_n\omega_n\omega)} \tag{20}$$

4 RESPONSE TO RANDOM LOADS

Next, the stochastic response of the coupled bending-torsion beam with supports and attached masses, subjected to stationary loads,will be obtainedusingthe frequency response functions derived in Section 3.

4.1 Response to concentrated loads

Consider the beam in Figure 1 subjected to a finite number K of stationary concentratedtransverse forces$P_r$applied at distance x_c from the elastic axis,assumedto be statistically dependent. The power spectral density functions of the deflection, $S_{HH}(y,\omega)$, andall response variables in vector $\mathbf{Y}(y)=\left[H \quad \Theta \quad M \quad S \quad \Psi \quad T\right]$, can be obtained by the following expressions involving the cross spectral density functions of the forces $S_{P_rP_s}(\omega)$

$$\begin{aligned}S_{Y_iY_i}(y,\omega)=\sum_{r=1}^{K}\sum_{s=1}^{K}\Big\{\Big[&Y_i^{(f)*}(y,y_r,\omega)Y_i^{(f)}(y,y_s,\omega)+Y_i^{(g)*}(y,y_r,\omega)Y_i^{(g)}(y,y_s,\omega)+\\&Y_i^{(f)*}(y,y_r,\omega)Y_i^{(g)}(y,y_s,\omega)+Y_i^{(g)*}(y,y_r,\omega)Y_i^{(f)}(y,y_s,\omega)\Big]S_{P_rP_s}(\omega)\Big\}\end{aligned} \tag{21}$$

In Eq.(21), the asterisk denotes complex conjugate and $Y_i^{(f)}(y,\xi,\omega),Y_i^{(g)}(y,\xi,\omega)$ are given as Eq.(15) and its analogous for $g(y)$,where $\mathbf{Y}^{(f)}(y)$ in Eq.(14) and $\mathbf{Y}^{(g)}(y)$ in Eq.(19) arereplaced respectively by $\mathbf{J}^{(P)}(y,\xi)$ and $\mathbf{J}^{(Mt)}(y,\xi)$.

4.2 Response to distributed loads

Assume now that the beam in Figure 1 is acted upon by a stationary distributed transverse load$f(t)$, applied at distance x_c from the elastic axis. The load israndomly varying with respect to time, but not spatially.If $S_{ff}(\omega)$ denotes the power spectral density function of $f(t)$, the power spectral density functions of all response variables can be obtained as follows:

$$S_{YY_i}(y,\omega)=\left[\left|Y_i^{(f)}(y,\omega)\right|^2+\left|Y_i^{(g)}(y,\omega)\right|^2+Y_i^{(f)*}(y,\omega)Y_i^{(g)}(y,\omega)+Y_i^{(g)*}(y,\omega)Y_i^{(f)}(y,\omega)\right]S_{ff}(\omega) \tag{22}$$

where $Y_i^{(f)}(y,\omega)$, $Y_i^{(g)}(y,\omega)$ are given respectively as Eq.(15)and its analogous for $g(y)$.

5 NUMERICAL EXAMPLE

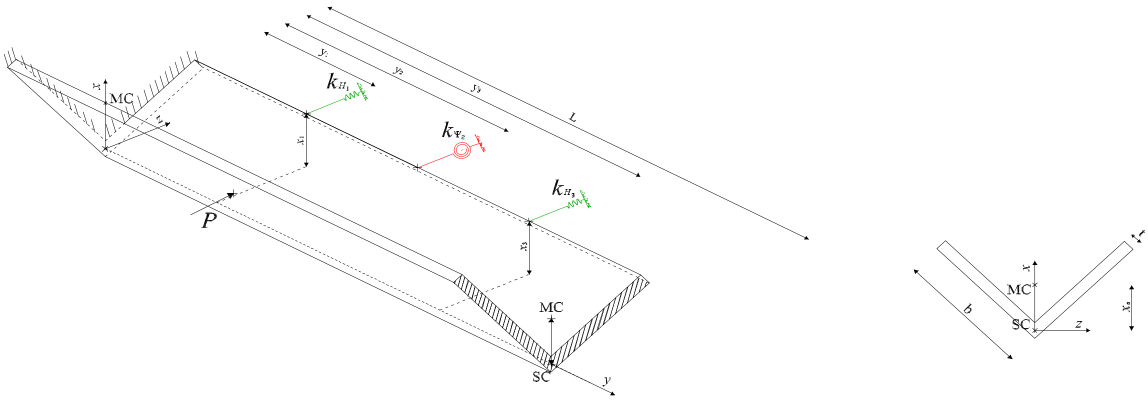

Figure 2: Clamped-clamped beam with angular section carrying elastic supports

Consider the clamped beam with angular cross section shown in Figure 2. Properties are chosen as follows:

$I = 1.0174 \cdot 10^{-5}\,\mathrm{m}^4, J = 9.6666 \cdot 10^{-8}, I_\alpha = 0.0549\,\mathrm{kg}\cdot\mathrm{m}, x_a = 0.0512\mathrm{m}, L = 3\mathrm{m}, m = 7.83\mathrm{kg}\cdot\mathrm{m}^{-1},$

$E = 70 \cdot 10^9\,\mathrm{N}\cdot\mathrm{m}^{-2}, G = 26.3158 \cdot 10^9\,N \cdot m^{-2}, c_1 = 5 \cdot 10^2\,\mathrm{Nm}^{-2}s, c_2 = 3.5 \cdot \mathrm{Ns}, b = 0.15\mathrm{m}, t = 0.01\mathrm{m}.$

The beam carries two translational elastic supports at locations $y_1 = 0.25L$ and $y_3 = 0.75L$, both applied at distance $x_1 = x_3 = 0.1025$ from the SC of the beam crosssection, and a torsional-rotational elastic support at the location $y_2 = 0.5L$ as shown in Figure 2.

The beam is acted upon by a stationary transverse load P with $S_{PP}(\omega) = 1$, applied at $y = 0.35L$ along the elastic axis parallel to $O - z$, at distance $x_c = x_a$ from y axis. Since the SC is eccentric with the respect to the MC along the x axis, as shown in Figure 2, the beam random response in the $y - z$ plane shall be investigated consideringthe coupling between bending and torsional vibrations.

Figure 3 shows the power spectral density functions of the bending deflection, $H(y)$, as well asthe deflection of the MC due to torsional rotation,$\Psi(y)x_a$, calculated at $y = 3L/7$ through the exact frequency response functionsbuiltin Section 3. Results are in excellent agreement. Figure 3 shows that twisting contributes significantly to the deflection of the MC, thus confirming the importance of accurate methods to capture bending-torsional coupling effects in the beam response.

6 CONCLUSIONS

The paper has presented exact analytical solutions for the response power spectral density functions of beams with mono-symmetric cross section, carrying an arbitrary number of in-span supports and attached masses, subjected to stationary loads. The solutions are obtained through two different closed-form expressions of the frequency response functions, using the theory of generalized functions and elementary coupled bending-torsion beam theory.Numerical results demonstrate the importance of bending-torsional coupling effects in beams under stationary loads, and the accuracy of the proposed approach.

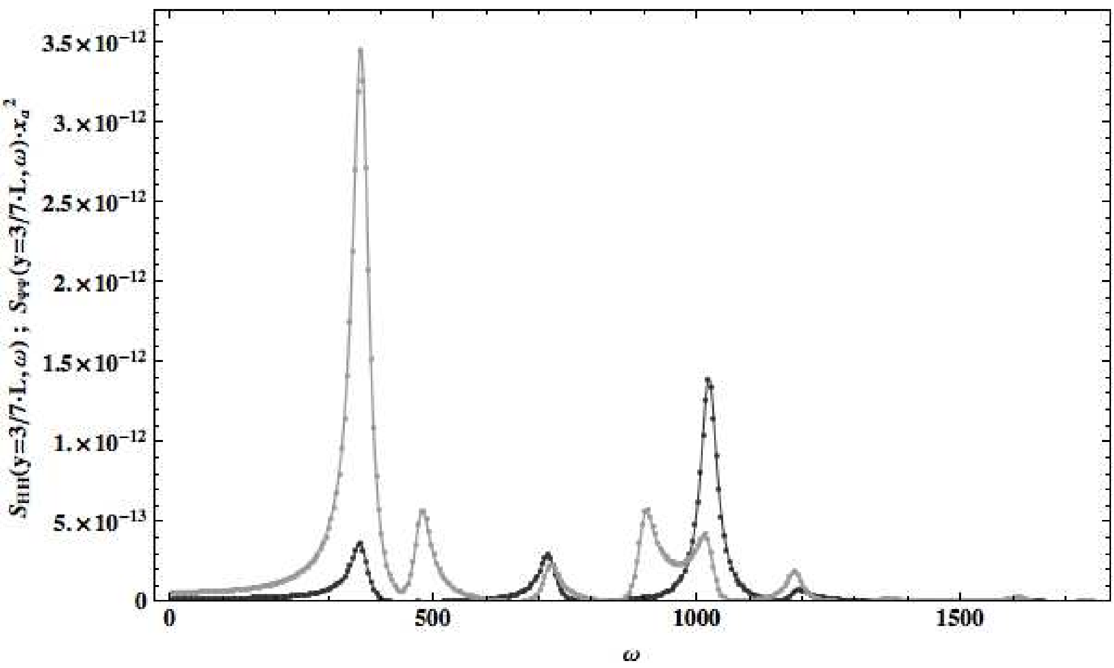

Figure 3:Power spectral densities of pure bending deflection $S_{HH}\left(y,\omega\right)$ (black), anddeflection of the MC due to torsional rotation $S_{\Psi\Psi}\left(y,\omega\right)x_a^2$ (gray), computed at $y=3L/7$, with frequency response functions given via generalized functions (continuous line) and normal mode method (dotted line).

BIBLIOGRAFY

[1] S. Di Lorenzo, M. Di Paola, F.P. Pinnola, A. Pirrotta, "Stochastic response of Fractionally damped beams.", *Prob. Eng. Mech.*, **35**, 37-43 (2014).

[2] J.R. Banerjee and D.Kennedy, "Response of an axially loaded Timoshenko beams to random loads.", *J. Sound Vib.*, **101 (4)**, 37-43 (1985).

[3] J.R. Banerjee, "Coupled bending-torsional dynamic stiffness matrix for beam elements.", *Int. J. Num. Method. Eng.*, **28**, 1283-1298 (1989).

[4] J.R. Banerjee, "Explicit frequency equation and mode shapes of a cantilever beam coupled in bending and torsion.", *Int. J. Num. Method. Eng.*, **28**, 1283-1298 (1989).

[5] S. Timoshenko, D.H. Young, W.J.R. Weaver, *Vibration Problems in Engineering*, Wiley,New York (1974).

[6] A.N. Bercin and M. Tanaka,"Coupled flexural-torsional vibrations of Timoshenko beams.", *J. Sound Vib.*, **207 (1),** 47-59(1997).

[7] S.H.R. Eslimy and J.R. Banerjee, "Use of generalized mass in the interpretation of dynamic response of bending-torsion coupled beams.", *J. Sound Vib.*, **238 (2)**, 295-308 (2000).

[8] S.H.R. Eslimy and J.R. Banerjee, "Response of a bending torsion coupled beam to deterministic and random loads.", *J. Sound Vib.*, **195 (2)**, 267-283 (1996).